The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

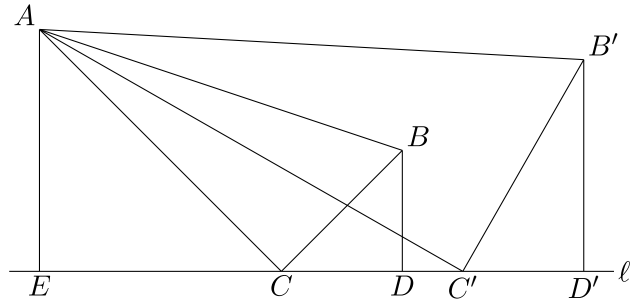
AMO TRAINING SESSIONS

2007 Senior Mathematics Contest: Problems 4 and 5 with Solutions

4. Let ABC and $AB'C'$ be similar right-angled triangles with right angle at C and C', respectively. Let ℓ be the line through C and C', and let D and D' be the points on ℓ such that BD and $B'D'$ are perpendicular to ℓ.

   Prove that $CD = C'D'$.

   Solution. We construct triangles ABC and $AB'C'$ as prescribed with line ℓ through C and C', and points D and D' on ℓ such that $BD \perp \ell$ and $B'D' \perp \ell$. Further, we produce A to a point E on ℓ such that $AE \perp \ell$.



Now, since

$$\angle AEC = \angle CDB = 90^\circ, \tag{1}$$

each of $\angle EAC$ and $\angle DCB$ is the complement of $\angle ACE$. So

$$\angle EAC = \angle DCB \tag{2}$$

$$\therefore \triangle AEC \sim \triangle CDB, \qquad \text{by (1) and (2), and the AA Rule}$$

$$\therefore \frac{AE}{AC} = \frac{CD}{CB}$$

$$CD = AE \times \frac{CB}{AC}.$$

By an identical argument with B', C', D' in place of B, C, D, respectively, we have

$$C'D' = AE \times \frac{C'B'}{AC'}.$$

Now, since $\triangle ABC \sim \triangle AB'C'$,

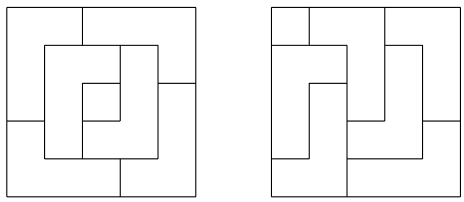
$$\frac{CB}{AC} = \frac{C'B'}{AC'}$$

$$\begin{aligned}\therefore CD &= AE \times \frac{CB}{AC}\\ &= AE \times \frac{C'B'}{AC'}\\ &= C'D'.\end{aligned}$$

5. An L-tetromino is a tile in the shape of a letter L, or of its mirror image, consisting of four unit squares as shown below: I have six L-tetrominoes of either shape and a 5×5 grid of unit squares. I can place six of these L-tetrominoes in the grid so that there is only one square of the grid not covered.

Where can that square be?

Solution. Firstly, the following coverings show that a corner square (via the second diagram or a rotation) or the middle square can be left uncovered by placing six L-tetrominoes on the 5×5 grid in such a way as to leave just one square uncovered.



We must show that a corner square or the middle square are the only possible squares that can be left uncovered by placing six L-tetrominoes on the 5×5 grid so that one square is left uncovered.

1. Colour the square in checkerboard fashion, so that the diagonals and middle squares on each edge are dark squares. Then each tetromino covers exactly two dark squares and two light squares. There is one more dark square (13 such squares) than light squares (12 such squares). So a dark square must be left uncovered. (This is called a *parity argument.*)
   Thus a square on a diagonal or on the middle of an edge must be the square left uncovered.
2. Now colour the columns of the square alternately dark and light, so that the outer and middle columns are dark. Each tetromino now covers either 3 dark squares and 1 light square or 3 light squares and 1 dark square.
   Suppose a light square is left uncovered and n tetrominoes cover 3 light squares. Then

$$3n + 6 - n = 10 - 1 \implies 2n = 3,$$

   which is impossible. So a light square cannot be left uncovered.
   This leaves the question of whether a middle square on an edge can be left uncovered.
3. If a centre square on an edge is not covered then all corner squares are covered. There are six ways to cover a corner square. In four of the ways a centre edge square is covered at the same time. In the remaining two ways a square second from the corner square in the same row or column is left uncovered and the only way to cover this square is to place a tetromino that also covers a centre edge square.
   Thus covering a corner square implies a centre edge square is also covered.
   So when 4 corner squares are covered, 4 centre edge squares are covered.

Thus the only squares that may be left uncovered are the corner squares and the middle square.