The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2008 Senior Mathematics Contest Problems

1. Let $a, b, c \in \mathbb{N}$. Prove that

$$\frac{bc}{a^2b+c} \leq \frac{b+c}{(1+a)^2}.$$

When does equality hold?

2. Let ABC be an acute-angled triangle, and let D be the point on AB (extended if necessary) such that $AB \perp CD$. Let t_A and t_B be the tangents to the circumcircle of $\triangle ABC$, through A and B, respectively. Let E and F be the points on t_A and t_B, respectively, such that $CE \perp t_A$ and $CF \perp t_B$.

Prove that

$$\frac{CD}{CE} = \frac{CF}{CD}.$$

3. Determine all odd $x \in \mathbb{N}$ for which there are $y, z \in \mathbb{N}$ satisfying both

   (i) $8x + (2y-1)^2 = z^2$ and

   (ii) $9 \leq 3(y+1) \leq x$.

4. Kate and Len play the following game using a heap of 2008 cards numbered

$$1, 2, 3, \ldots, 2008.$$

Len draws ℓ cards from the heap, records all the numbers he has drawn and then returns the cards to the heap. Then Kate draws k cards from the heap and records all the numbers she has drawn. Finally, they calculate all the non-zero differences between pairs of numbers they have recorded. Kate makes a list her differences ΔK, while Len's list of differences is ΔL.

   (a) Prove that ΔK and ΔL have at least one natural number in common if $k\ell \geq 4015$.

   (b) If $k\ell = 4014$, must ΔK and ΔL have a natural number in common? Give reasons for your answer.

5. Let c be a circle with centre O, T a point on c, t the tangent to c at T and P a point on t other than T. Let ℓ be the line through P and O, and let Q be the point, other than P, on ℓ such that $QO = OP$. A line through Q intersects the circle in the two different points U and V, and TU and TV intersect ℓ in R and S, respectively.

Prove that $RO = SO$.