The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2008 Senior Mathematics Contest Problems 1, 3 and 4 with Solutions

1. Let $a, b, c \in \mathbb{N}$. Prove that

$$\frac{bc}{a^2b+c} \leq \frac{b+c}{(1+a)^2}.$$

When does equality hold?

Solution. Working backwards leads us to finding one of the following solutions.
Method 1.

$$\begin{aligned}
(ab-c)^2 &\geq 0 && (1)\\
a^2b^2+c^2-2abc &\geq 0\\
a^2b^2+c^2 &\geq 2abc && (2)\\
a^2b^2+c^2+bc+a^2bc &\geq 2abc+bc+a^2bc\\
(a^2b+c)(b+c) &\geq bc(a^2+2a+1)\\
(a^2b+c)(b+c) &\geq bc(a+1)^2\\
\frac{b+c}{(a+1)^2} &\geq \frac{bc}{a^2b+c}, \qquad \text{since } a,b,c>0 \implies (a+1)^2>0 \text{ and } a^2b+c>0\\
\therefore \frac{bc}{a^2b+c} &\leq \frac{b+c}{(1+a)^2}.
\end{aligned}$$

At the step (1), we have equality if and only if $ab-c=0$, i.e. $ab=c$.

Method 2. By AM-GM, since $a, b, c > 0$,

$$\begin{aligned}
\frac{\frac{c}{ab}+\frac{ab}{c}}{2} &\geq \sqrt{\frac{c}{ab}\cdot\frac{ab}{c}} = 1 && (3)\\
\frac{c}{ab}+\frac{ab}{c} &\geq 2\\
c^2+(ab)^2 &\geq 2abc
\end{aligned}$$

and the rest is as above from step (2).
At step (3), we have equality if and only if

$$\begin{aligned}
&\frac{c}{ab} = \frac{ab}{c}\\
\iff\ & c^2 = (ab)^2\\
\iff\ & c = ab, \qquad \text{since } a,b,c>0.
\end{aligned}$$

3. Determine all odd $x \in \mathbb{N}$ for which there are $y, z \in \mathbb{N}$ satisfying both

(i) $8x + (2y-1)^2 = z^2$ and

(ii) $9 \le 3(y+1) \le x$.

Solution. The key steps are:

- Show x not prime.
- Show x can be any composite positive odd integer.

Note: 9 is the smallest composite positive odd integer ($3, 5, 7$ are prime!).

By (i):

$$\begin{aligned} 8x &= z^2 - (2y-1)^2 \\ &= \big(z - (2y-1)\big)\big(z + (2y-1)\big) \end{aligned}$$

Since $2 \,\big|\, 8x$,

$$\begin{aligned} &2 \,\big|\, (z-2y+1)(z+2y-1) \\ \Longrightarrow\ &2 \,\big|\, (z-2y+1) \quad\text{or}\quad 2 \,\big|\, (z+2y-1) \\ \Longrightarrow\ &2 \,\big|\, (z-2y+1) \text{ and } 2 \,\big|\, (z+2y-1), \quad \text{since } z-2y+1 \equiv z+2y-1 \pmod 2 \end{aligned}$$

Now we show that x is not prime, and note that since $y, z \in \mathbb{N}$ we have $2y-1 \in \mathbb{N}$ so that $z-2y+1 < z+2y-1$. Suppose, for a contradiction, that x is prime. Then

Case 1: $4 \,\big|\, z+2y-1$. Then

$$\begin{aligned} z-2y+1 &= 2 && (4)\\ z+2y-1 &= 4x && (5)\\ \therefore 4y-2 &= 4x-2, && (5)-(4)\\ y &= x \\ \therefore 3(x+1) &\le x, && \text{by (ii)}\\ 2x &\le -3 \\ \therefore x &< 0, && \text{contradicts } x \in \mathbb{N}. \end{aligned}$$

Case 2: $4 \,\big|\, z-2y+1$. Then

$$\begin{aligned} z-2y+1 &= 4 && (6)\\ z+2y-1 &= 2x && (7)\\ \therefore 4y-2 &= 2x-4 \\ y &= \tfrac{1}{2}(x-1) \\ \therefore 3(\tfrac{1}{2}x - \tfrac{3}{2}) &\le x, && \text{by (ii)}\\ x &\le 9, && \text{but } x \ge 9 \text{ by (ii)}\\ \therefore x &= 9, && \text{contradicts assumption that } x \text{ is prime.} \quad (8) \end{aligned}$$

Case 3: $x \,\big|\, z-2y+1$. Then $x \ge 3$, since x is an odd prime. Thus

$$\begin{aligned} z-2y+1 &\ge 2x \\ &\ge 6 > 4 \ge z+2y-1 \end{aligned}$$

which contradicts $z-2y+1 < z+2y-1$.

$\therefore x$ is not prime, i.e. x is an odd composite natural number.
Now we show there are no other restrictions, by showing that all such numbers x satisfy both (i) and (ii).
If $x \in \mathbb{N}$ such that it is odd and composite, then it is of form:

$$x = (2v-1)w, \text{ where } v \geq 2, w \geq 2v-1.$$

Put

$$y = w - v + 1 \ (\geq v \geq 2 \implies y \in \mathbb{N})$$

Also, $y \leq w - 1$, since $v \geq 2$. Then

$$\begin{aligned}
2 \leq v \leq y &\leq w - 1 \\
&= \frac{x}{2v-1} - 1 \\
&\leq \frac{x}{3} - 1, \qquad \text{since } 2v-1 \geq 3 \\
&= \frac{x-3}{3} \\
\therefore 2 \leq y &\leq \frac{x-3}{3} \\
6 \leq 3y &\leq x - 3 \\
\therefore 9 \leq 3(y+1) &\leq x,
\end{aligned}$$

and so (ii) is satisfied.
Now, we check (i):

$$\begin{aligned}
8x + (2y-1)^2 &= 8(2v-1)w + \big(2(w-v+1)-1\big)^2 \\
&= \big(2w-(2v-1)\big)^2 + 4\cdot 2w(2v-1) \\
&= \big(2w+(2v-1)\big)^2.
\end{aligned}$$

Thus, if we put $z = 2w + 2v - 1$ which we note is odd and positive, then (i) is satisfied.
Thus the triple $(x, y, z) = \big((2v-1)w, w-v+1, 2w+2v-1\big)$, satisfies (i) and (ii) for all $v, w \in \mathbb{N}$ such that $v \geq 2$ and $w \geq 2v-1$, which shows that x can be any positive odd composite integer.

4. Kate and Len play the following game using a heap of 2008 cards numbered

$$1, 2, 3, \ldots, 2008.$$

Len draws ℓ cards from the heap, records all the numbers he has drawn and then returns the cards to the heap. Then Kate draws k cards from the heap and records all the numbers she has drawn. Finally, they calculate all the non-zero differences between pairs of numbers they have recorded. Kate makes a list her differences ΔK, while Len's list of differences is ΔL.

(a) Prove that ΔK and ΔL have at least one natural number in common if $k\ell \geq 4015$.

(b) If $k\ell = 4014$, must ΔK and ΔL have a natural number in common? Give reasons for your answer.

Solution.

(a) Let K be the set of numbers drawn by Kate, let L be the set of numbers drawn by Len, and let

$$S = \{a + b \mid a \in K, b \in L\}.$$

Note that for any $s \in S$,

$$2 = 1 + 1 \leq s \leq 2008 + 2008 = 4016.$$

Hence, the largest S can be is $\{2, 3, \ldots, 4016\}$.
The set $K \times L$ is defined to be the set of all ordered pairs (a, b) where the first coordinate a is in K and the second coordinate b is in L, i.e.

$$K \times L = \{(a, b) \mid a \in K, b \in L\}.$$

So another way of representing S is

$$S = \{a + b \mid (a, b) \in K \times L.$$

This means we have a *function* f (which just adds coordinates) as follows,

$$\begin{array}{rccc} f: & K \times L & \to & \{2, 3, \ldots, 4016\} \\ & (a, b) & \mapsto & a + b \end{array}$$

where

$$\begin{aligned} \text{domain}(f) &= K \times L \text{ and} \\ \text{range}(f) &= S \subseteq \{2, 3, \ldots, 4016\}. \end{aligned}$$

For any function, the cardinality of its range cannot be larger than the cardinality of its domain, i.e.

$$|\text{domain}(f)| \geq |\text{range}(f)|,$$

with equality when f is one-to-one. (This is a fairly trivial application of the Pigeon Hole Principle.)
In our case,

$$\begin{aligned} kl = |K \times L| &= |\text{domain}(f)| \\ &\geq |\text{range}(f)| = |S| \\ |S| &\leq |\{2, 3, \ldots, 4016\}| = 4015. \end{aligned}$$

Case 1: $S = \{2, 3, \ldots, 4016\}$. Since $2 \in S$, we must have 1 in both K and L. Similarly, since $4016 \in S$, we must have 2008 in both K and L. Since 1 and 2008 are in both K and L, $2008 - 1 = 2007$ is in both ΔK and ΔL. So ΔK and ΔL have an element in common in this case.

Case 2: $S \neq \{2, 3, \ldots, 4016\}$. Then $|S| < 4015$ while $k\ell = |K \times L| \geq 4015$. Since $|K \times L| > |S|$, the function f is not one-to-one. Thus there must be two elements $(a, b), (a', b') \in K \times L$ with the same sum, i.e. there are $(a, b), (a', b') \in K \times L$ such that

$$\begin{aligned} a + b &= a' + b' \\ \therefore a - a' &= b' - b \\ |a - a'| &= |b - b'| \end{aligned}$$

so that ΔK and ΔL again have a common element.

(b) Suppose $K = \{1, 2008\}$, so that $\Delta K = \{2007\}$ and $L = \{1, 2, \ldots, 2007\}$, so that $\Delta L = \{1, 2, \ldots, 2006\}$. Then $k\ell = 2 \cdot 2007 = 4014$ and $\Delta K \cap \Delta L = \emptyset$. Thus if $k\ell = 4014$, ΔK and ΔL need not have a natural number in common.