The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

**AMO TRAINING SESSIONS**

# 2010 Senior Mathematics Contest Problems

1. Let $ABCD$ be a convex quadrilateral. Let $X$ be a point on diagonal $BD$ such that $AX \perp BC$, and let $Y$ be a point on $AC$ such that $DY \perp AD$. Suppose $XY \parallel BC$.

   Prove that $ABCD$ is cyclic.

2. Determine all real-valued functions $f$ that satisfy

$$2f(xy + xz) + 2f(xy - xz) \geq 4f(x)f(y^2 - z^2) + 1$$

   for all real numbers $x, y, z$.

3. For $n \in \mathbb{N}$, define

$$S(n) = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor,$$

   where $\lfloor x \rfloor$ is the greatest integer not exceeding $x$.

   Prove that $S(n+1) - S(n) = 2$ if and only if $n+1$ is a prime number.

4. Given any nine integers, prove that it is possible to choose five of them such that their sum is divisible by 5.

5. Outside $\triangle ABC$, points $A', B', C'$ are chosen such that $\triangle ABC'$, $\triangle BCA'$ and $\triangle CAB'$ are external to $\triangle ABC$ and $\angle AC'B + \angle BA'C + \angle CB'A = 180°$. Let $O_A, O_B, O_C$ be the circumcentres of $\triangle A'BC$, $\triangle B'CA$, $\triangle C'AB$, respectively.

   Prove that $\angle O_A O_C O_B = \angle AC'B$.