The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

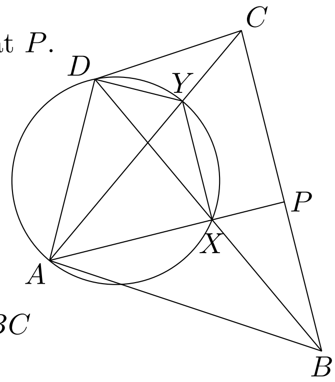
AMO TRAINING SESSIONS

2010 Senior Mathematics Contest Problems with Some Solutions

1. Let $ABCD$ be a convex quadrilateral. Let X be a point on diagonal BD such that $AX \perp BC$, and let Y be a point on AC such that $DY \perp AD$. Suppose $XY \parallel BC$.

   Prove that $ABCD$ is cyclic.

   Solution. Drop a perpendicular from A to BC, to meet BC at P. Let X be the point of intersection of AP and BD. Then X is the point on the diagonal BD of quadrilateral $ABCD$ such that $AX \perp BC$.
   Now construct the perpendicular from AD to meet AC at Y. Then Y is the point on AC such that $DY \perp AD$.
   Suppose also that $XY \parallel BC$. Then



$$\begin{aligned}
AX &\perp XY, && \text{since } XY \parallel BC \text{ and } AX \perp BC\\
DY &\perp AD\\
\therefore \angle YDA &= 90^\circ = \angle YXA\\
\therefore \angle YDA, \angle YXA &\text{ are supplementary}\\
\therefore AXYD &\text{ is cyclic}\\
\therefore \angle ADB &= \angle ADX, && \text{(same angle)}\\
&= \angle AYX, && \text{(starting on same arc } AB\text{)}\\
&= \angle ACP, && \text{(corresponding angles, since } XY \parallel BC\text{)}\\
&= \angle ACB, && \text{(same angle)}\\
\therefore \angle ADB, \angle ACB &\text{ stand on an arc of a circle}\\
\therefore ABCD &\text{ is cyclic.}
\end{aligned}$$

2. Determine all real-valued functions f that satisfy

$$2f(xy + xz) + 2f(xy - xz) \geq 4f(x)f(y^2 - z^2) + 1$$

   for all real numbers x, y, z.

   Solution.

3. For $n \in \mathbb{N}$, define

$$S(n) = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor,$$

   where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

   Prove that $S(n+1) - S(n) = 2$ if and only if $n+1$ is a prime number.

4. Given any nine integers, prove that it is possible to choose five of them such that their sum is divisible by 5.

Solution. We work modulo 5. For convenience we will use the symmetric representation of the residue classes mod 5 to be $0, \pm 1, \pm 2$. Suppose we are given nine integers.

Case 1: There are 5 numbers that belong to the same residue class, a say, mod 5. Then the sum of these 5 numbers is congruent to

$$5a \equiv 0 \pmod 5.$$

Case 2: Each residue class has at least one member. Then choose one number from each of the classes; their sum is congruent to

$$0 + 1 + -1 + 2 + -2 \equiv 0 \pmod 5.$$

Case 3: Neither Case 1, nor Case 2 occurs. Since Case 2 does not occur, the nine numbers belong to at most four of the classes, and so by the Pigeon Hole Principle, there is a class with three members.

Now observe that we can add a constant k to all the numbers without changing the problem (the total of any five of the numbers varies by $5k$ and so the sum is unaltered mod 5).

Thus we can assume the class with three members is the 0 class. Since Case 1 does not occur there are at most four numbers in the 0 class, and hence at least five of the numbers are non-zero mod 5; let these five numbers be x_1, x_2, x_3, x_4, x_5.

Now form the partial sums:

$$\begin{aligned}
S_0 &= 0 \\
S_1 &= x_1 \\
S_2 &= x_1 + x_2 \\
S_3 &= x_1 + x_2 + x_3 \\
S_4 &= x_1 + x_2 + x_3 + x_4 \\
S_5 &= x_1 + x_2 + x_3 + x_4 + x_5
\end{aligned}$$

By the Pigeon Hole Principle again, since there are five residue classes and six partial sums, at least two of these sums S_i, S_j, (with $j > i$) say, belong to the same residue class.

Now $j \neq i + 1$, since this implies $x_j \equiv 0 \pmod 5$, contrary to assumption. Hence $j \geq i + 2$. So we have

$$\begin{aligned}
x_1 + \cdots + x_i &\equiv x_1 + \cdots + x_i + x_{i+1} + \cdots + x_j \pmod 5 \\
\therefore 0 &\equiv x_{i+1} + \cdots + x_j \pmod 5
\end{aligned}$$

This last sum has at least two members. Take whatever of the three members of the 0 class we need to make up five numbers. Then the total of these five numbers is congruent to 0 mod 5.

Thus in all cases, five numbers can be chosen from the nine numbers, such that their total is divisible by 5 (since the total being congruent to 0 mod 5 implies the total is divisible by 5).

5. Outside $\triangle ABC$, points A', B', C' are chosen such that $\triangle ABC'$, $\triangle BCA'$ and $\triangle CAB'$ are external to $\triangle ABC$ and $\angle AC'B + \angle BA'C + \angle CB'A = 180°$. Let O_A, O_B, O_C be the circumcentres of $\triangle A'BC$, $\triangle B'CA$, $\triangle C'AB$, respectively.

   Prove that $\angle O_A O_C O_B = \angle AC'B$.