The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2011 Senior Mathematics Contest Problems

1. Determine all pairs (x, y) of integers that satisfy

$$(x + y + 11)^2 = x^2 + y^2 + 11^2.$$

2. Suppose $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f(x)y = f\big(xf(y)\big), \ \ \forall\, x, y > 0.$$

   Prove that $f(xy) = f(x)f(y) \ \ \forall\, x, y > 0$.

3. Let P be a point inside $\triangle ABC$. The line through A and P intersects the circumcircle of $\triangle ABC$, at points A and A_1. The point A_2 is the midpoint of line segment AA_1. The points B_2 and C_2 are defined similarly.

   Prove that P lies on the circumcircle of $\triangle A_2B_2C_2$.

4. Let $p(x)$ be a polynomial with integer coefficients. Let m, n be distinct integers such that $p(m)p(n) = -(m - n)^2$.

   Prove that $p(m) + p(n) = 0$.

5. In a group of 2011 people of different heights, Zelda is the 27^{th} tallest person. Let n be the number of ways that this group can form a queue such that Zelda is shorter than everyone ahead of her in the queue.

   Prove that there is exactly one set S of 2010 distinct positive integers such that n is the product of elements of S.