The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2013 Senior Mathematics Contest Problems

1. For any two finite sets A and B, define $f(A, B)$ to be the number of elements that are contained in either A or in B, but not in both A and B.
   Three given sets X, Y, Z satisfy
   $$f(X, Y) = f(Y, Z) = f(Z, X).$$
   (a) Prove that $f(X, Y)$ is even.
   (b) Find a set W such that
   $$f(W, X) = f(W, Y) = f(W, Z) = \tfrac{1}{2} f(X, Y).$$

2. Let A, B, C, D, E, F, G be different points in the plane such that
   $$AB = BC = CD = DE = EF = FA = AG = CG = EG.$$
   Prove that lines AD, BE and CF concur.

3. Prove that there are no positive integers m, n such that both $4m^2 + 17n^2$ and $17m^2 + 4n^2$ are perfect squares.

4. Let $ABCD$ be a convex quadrilateral such that $AB = AD$ and $CB = CD$.
   Let E be the point on line CD such that $AE \parallel BC$.
   Let lines BD and AE intersect at G.
   Let F be the point on line BC such that $FG \parallel CE$.
   Prove that BE passes through the midpoint of AF.

5. A sequence of polynomials $p_0(x), p_1(x), p_2(x), \ldots$ are defined by
   $$\begin{aligned} &p_0(x) = 0, \\ &p_1(x) = x - 2013, \\ &p_n(x) = (x - 2013)p_{n-1}(x) + (2014 - x)p_{n-2}(x), \quad \text{for } n \geq 2. \end{aligned}$$
   For each $n \in \mathbb{N}$, determine all real solutions of $p_n(x) = 0$.