AMOC SENIOR CONTEST

Tuesday, 12 August 2014
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Each point in the plane is labelled with a real number. For each cyclic quadrilateral $ABCD$ in which the line segments AC and BD intersect, the sum of the labels at A and C equals the sum of the labels at B and D.

Prove that all points in the plane are labelled with the same number.

2. For which integers $n \geq 2$ is it possible to separate the numbers $1, 2, \ldots, n$ into two sets such that the sum of the numbers in one of the sets is equal to the product of the numbers in the other set?

3. Consider functions f defined for all real numbers and taking real numbers as values such that

$$f(x+14) - 14 \leq f(x) \leq f(x+20) - 20, \qquad \text{for all real numbers } x.$$

Determine all possible values of $f(8765) - f(4321)$.

4. Let ABC be a triangle such that $\angle ACB = 90°$. The point D lies inside triangle ABC and on the circle with centre B that passes through C. The point E lies on the side AB such that $\angle DAE = \angle BDE$. The circle with centre A that passes through C meets the line through D and E at the point F, where E lies between D and F.

Prove that $\angle AFE = \angle EBF$.

5. Ada tells Byron that she has drawn a rectangular grid of squares and placed either the number 0 or the number 1 in each square. Next to each row, she writes the sum of the numbers in that row. Below each column, she writes the sum of the numbers in that column. After Ada erases all of the numbers in the squares, Byron realises that he can deduce each erased number from the row sums and the column sums.

Prove that there must have been a row containing only the number 0 or a column containing only the number 1.