1. Each point in the plane is labelled with a real number. For each cyclic quadrilateral $ABCD$ in which the line segments AC and BD intersect, the sum of the labels at A and C equals the sum of the labels at B and D.

   Prove that all points in the plane are labelled with the same number.

Solution 1 (Angelo Di Pasquale)

For any point P in the plane, let $f(P)$ denote its label. Consider two points A and B, and construct any cyclic pentagon $ABPQR$ whose vertices lie in that order.

Then we have $f(A) + f(Q) = f(P) + f(R) = f(B) + f(Q)$.

This implies that the arbitrarily chosen points A and B satisfy $f(A) = f(B)$.

So it is necessarily true that all points in the plane are labelled with the same number.

Solution 2 (Chaitanya Rao)

For any point P in the plane, let $f(P)$ denote its label. Take an arbitrary cyclic quadrilateral $ABCD$, where AC and BD intersect, and consider points on the circumcircle. Note that every point P on the circular arc DB containing A satisfies

$$f(P) = f(B) + f(D) - f(C) = f(A) + f(C) - f(C) = f(A).$$

Similarly, every point Q on the circular arc AC containing B satisfies

$$f(Q) = f(A) + f(C) - f(D) = f(B) + f(D) - f(D) = f(B).$$

We can choose $P = Q$ on the arc AB not containing C or D in order to deduce that $f(A) = f(B)$.

Since the points A and B can be chosen arbitrarily, it follows that all points in the plane are labelled with the same number.

2. For which integers $n \geq 2$ is it possible to separate the numbers $1, 2, \ldots, n$ into two sets such that the sum of the numbers in one of the sets is equal to the product of the numbers in the other set?

Solution 1 (Angelo Di Pasquale)

Suppose that x, y, z are in one of the groups and that the rest of the numbers are in the other group. This information leads to the equation

$$xyz = 1 + 2 + \cdots + n - x - y - z = \frac{n(n+1)}{2} - x - y - z.$$

If we substitute $z = 1$ in the equation above, we can rearrange to obtain

$$(x+1)(y+1) = \frac{n(n+1)}{2}.$$

If n is even, we can take $x = \frac{n}{2} - 1$ and $y = n$.

If n is odd, we can take $x = \frac{n-1}{2}$ and $y = n - 1$.

These constructions can be carried out as long as x, y, z are all different, which holds for $n \geq 5$. It is easy to check that the task is impossible for $n = 2$ and $n = 4$. On the other hand, we have $1 + 2 = 3$ for $n = 3$. Therefore, the task is possible only for $n = 3$ and all integers $n \geq 5$.

Solution 2 (Daniel Mathews and Kevin McAvaney)

By examining small values of n, one can directly observe the following patterns and verify them.

- If $n = 2k$ for an integer $k \geq 3$, we have

$$[1 + 2 + \cdots + (2k)] - 1 - (k-1) - (2k) = 1 \times (k-1) \times (2k).$$

  This fact follows from the identity $1 + 2 + \cdots + (2k) = k(2k+1)$.

- When $n = 2k + 1$ for an integer $k \geq 2$, we have

$$[1 + 2 + \cdots + (2k+1)] - 1 - k - (2k) = 1 \times k \times (2k).$$

  This fact follows from the identity $1 + 2 + \cdots + (2k+1) = (k+1)(2k+1)$.

The remaining cases $n = 1, 2, 3, 4$ can be handled individually, as in the previous solution. Therefore, the task is possible only for $n = 3$ and all integers $n \geq 5$.

3. Consider functions f defined for all real numbers and taking real numbers as values such that

$$f(x+14) - 14 \leq f(x) \leq f(x+20) - 20, \qquad \text{for all real numbers } x.$$

Determine all possible values of $f(8765) - f(4321)$.

Solution 1

Replace x by $x - 14$ in the left inequality and x by $x - 20$ in the right inequality to obtain

$$f(x-20) + 20 \leq f(x) \leq f(x-14) + 14.$$

It follows by induction that the following inequalities hold for every positive integer n.

$$\begin{aligned} f(x-20n) + 20n \leq f(x) \leq f(x-14n) + 14n \\ f(x+14n) - 14n \leq f(x) \leq f(x+20n) - 20n \end{aligned}$$

Therefore, we have the following chains of inequalities.

$$\begin{aligned} f(x+2) &= f(x + 20 \times 5 - 14 \times 7) \\ &\geq f(x + 20 \times 5) - 98 \\ &\geq f(x) + 100 - 98 \\ &\geq f(x) + 2 \end{aligned} \qquad \begin{aligned} f(x+2) &= f(x + 14 \times 3 - 20 \times 2) \\ &\leq f(x + 14 \times 3) - 40 \\ &\leq f(x) + 42 - 40 \\ &\leq f(x) + 2 \end{aligned}$$

So we have deduced that $f(x+2) = f(x) + 2$ and it follows by induction that

$$f(x+2n) = f(x) + 2n$$

for all real numbers x and all positive integers n.

Substituting $x = 4321$ and $n = 2222$ into this equation yields

$$f(8765) - f(4321) = 4444.$$

Since $f(x) = x$ satisfies the conditions of the problem, the only possible value of the expression $f(8765) - f(4321)$ is 4444.

Solution 2 (Angelo Di Pasquale)

Taking advantage of the fact that $140 = 7 \times 20 = 10 \times 14$, we have the following chains of inequalities.

$$f(x) \le f(x+20) - 20 \le f(x+40) - 40 \le \cdots \le f(x+140) - 140$$
$$f(x) \ge f(x+14) - 14 \ge f(x+28) - 28 \ge \cdots \ge f(x+140) - 140$$

So by the squeeze principle, equality holds throughout and we have $f(x) = f(x+20) - 20$ and $f(x) = f(x+14) - 14$ for all real numbers x.

Then since $8765 - 4405 = 4360$ is a multiple of 20 and $4405 - 4321 = 84$ is a multiple of 14, it follows that

$$f(8765) - f(4405) = 8765 - 4405 \quad \text{and} \quad f(4405) - f(4321) = 4405 - 4321.$$

Adding these equations yields $f(8765) - f(4321) = 8765 - 4321 = 4444$.

Since $f(x) = x$ satisfies the conditions of the problem, the only possible value of the expression $f(8765) - f(4321)$ is 4444.

Solution 3 (Joe Kupka)

We have the following chain of inequalities for all real numbers x.

$$\begin{aligned} \cdots \le f(x+28) - 28 \le f(x+14) - 14 \le f(x) \\ \le f(x+20) - 20 \le f(x+40) - 40 \le \cdots \end{aligned}$$

Using $f(x+42) - 42 \le f(x+40) - 40$ yields $f(x+2) \le f(x) + 2$.

It follows that $f(x+4) \le f(x) + 4$ for all real numbers x.

Similarly, using $f(x+56) - 56 \le f(x+60) - 60$ yields $f(x+4) \ge f(x) + 4$ for all real numbers x.

It follows now that $f(x+4) = f(x) + 4$ and by induction, $f(x+4n) = f(x) + 4n$ for all real numbers x and positive integers n.

Setting $x = 4321$ and $n = 1111$ gives $f(8765) - f(4321) = 4444$.

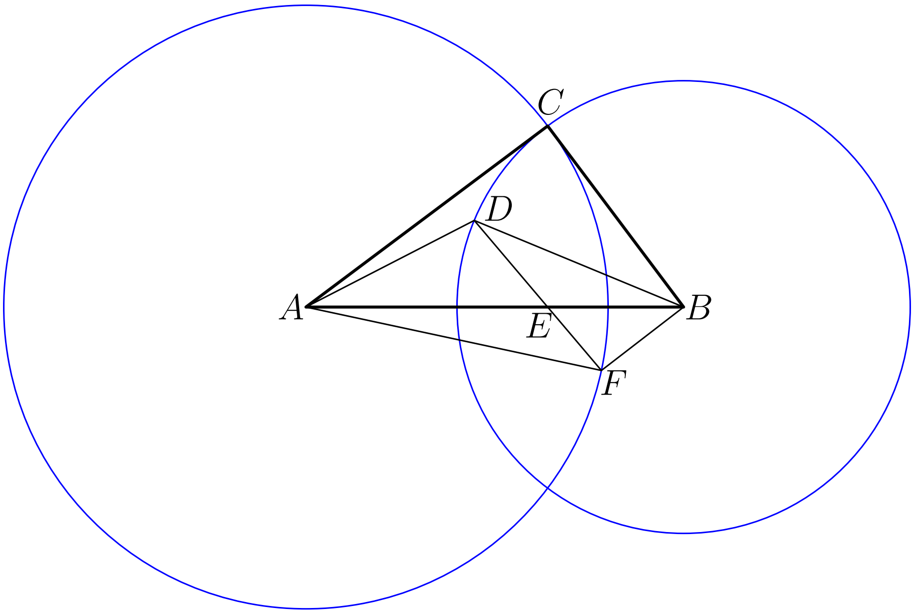
Since $f(x) = x$ satisfies the conditions of the problem, the only possible value of the expression $f(8765) - f(4321)$ is 4444.

4. Let ABC be a triangle such that $\angle ACB = 90°$. The point D lies inside triangle ABC and on the circle with centre B that passes through C. The point E lies on the side AB such that $\angle DAE = \angle BDE$. The circle with centre A that passes through C meets the line through D and E at the point F, where E lies between D and F.

Prove that $\angle AFE = \angle EBF$.

Solution 1

Since $\angle BAD = \angle BDE$ by assumption and $\angle DBA = \angle EBD$, we know that triangles BAD and BDE are similar.



Hence, we have the equal ratios

$$\frac{EB}{DB} = \frac{DB}{AB} \quad \Rightarrow \quad EB = \frac{DB^2}{AB}.$$

Therefore, we can deduce the following sequence of equalities.

$$\begin{aligned}
AE &= AB - EB \\
&= AB - \frac{BD^2}{AB} \\
&= \frac{AB^2 - BD^2}{AB} \\
&= \frac{AB^2 - BC^2}{AB} && (D \text{ lies on the circle with centre } B \text{ through } C) \\
&= \frac{AC^2}{AB} && (\text{Pythagoras' theorem in triangle } ABC) \\
&= \frac{AF^2}{AB} && (F \text{ lies on the circle with centre } A \text{ through } C)
\end{aligned}$$

This implies that $\frac{AF}{AB} = \frac{AE}{AF}$, which combines with the fact that $\angle BAF = \angle FAE$ to show that triangles AFB and AEF are similar.

Therefore, we conclude that

$$\angle AFE = \angle ABF = \angle EBF.$$

Solution 2 (Angelo Di Pasquale)

Since we know that $\angle DAE = \angle BDE$, it follows from the alternate segment theorem that BD is tangent to the circumcircle of triangle ADE at D.

So the power of a point theorem implies that

$$BD^2 = AB \cdot BE \quad \Rightarrow \quad BC^2 = AB \cdot (AB - AE) = AB^2 - AB \cdot AE.$$

The second equation follows from the first since $BC = BD$ and $BE = AB - AE$.

By Pythagoras' theorem, we know that $AC^2 + BC^2 = AB^2$. Combining this with the previous equation and the fact that $AC = AF$, we deduce that

$$AB \cdot AE = AC^2 \quad \Rightarrow \quad AB \cdot AE = AF^2.$$

By the power of a point theorem, this implies that AF is tangent to the circumcircle of triangle BEF at F.

Now invoke the alternate segment theorem to conclude that $\angle AFE = \angle EBF$.

5. Ada tells Byron that she has drawn a rectangular grid of squares and placed either the number 0 or the number 1 in each square. Next to each row, she writes the sum of the numbers in that row. Below each column, she writes the sum of the numbers in that column. After Ada erases all of the numbers in the squares, Byron realises that he can deduce each erased number from the row sums and the column sums.

   Prove that there must have been a row containing only the number 0 or a column containing only the number 1.

Solution 1

Call a rectangular array of numbers *amazing* if each number in the array is equal to 0 or 1 and no other array has the same row sums and column sums. We are required to prove that in an amazing array, there must be a row containing only the number 0 or a column containing only the number 1.

Call the four entries in the intersection of two rows and two columns a *rectangle*. We say that a rectangle is *forbidden* if

- its top-left and bottom-right entries are 0, while its top-right and bottom-left entries are 1; or
- its top-left and bottom-right entries are 1, while its top-right and bottom-left entries are 0.

It should be clear that an amazing array cannot have forbidden rectangles, since switching 0 for 1 and vice versa in a forbidden rectangle will produce another array with the same row sums and column sums.

Suppose that there exists an amazing array in which there is no row containing only the number 0 nor a column containing only the number 1. Let row a have the maximum number of entries equal to 0. Then row a contains the number 1, in column m, say. Furthermore, this column contains the number 0, in row b, say. In order to avoid a forbidden rectangle, row b must have a 0 in every column in which row a has a 0.

Since row b also has a 0 in column m, while row a has a 1 in column m, this contradicts the fact that row a has the maximum number of entries equal to 0.

Therefore, every amazing array has a row containing only the number 0 or a column containing only the number 1.

Solution 2 (Ivan Guo and Alan Offer)

We will use the notion of an amazing array, defined in Solution 1.

Suppose that every row contains the number 1 and every column contains the number 0 in an amazing array. So for an entry equal to 0 in the array, we can find an entry equal to 1 in its row. Then for that entry equal to 1, we can find an entry equal to 0 in its column, and so on.

Eventually, we must return to an entry already considered. At this point, we have identified a sequence of distinct squares in the array

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n,$$

such that a_i contains a 0 and b_i contains a 1 for all i. Furthermore, a_i and b_i are in the same row, while b_i and a_{i+1} are in the same column for all i, where we take $a_{n+1} = a_1$.

After switching the entry in each square a_i from 0 to 1 and the entry in each square b_i from 1 to 0, we obtain another array with identical row and column sums. This contradicts the fact that the array is amazing.

Solution 3 (Daniel Mathews)

We will use the notion of an amazing array, defined in Solution 1.

Lemma. If A is an amazing array, and B is obtained from A by removing a row or a column, then B is also amazing.

Proof. If B is not amazing, then there are two distinct arrays B, B' with the same row and column sums. We can then add in the removed row or column to obtain two distinct arrays A, A' with the same row and column sums, contradicting the fact that A is amazing. $\square$

We now use this lemma to prove the desired result.

Suppose that A is an amazing array in which every row has a 1 and every column has a 0, in order to derive a contradiction. We claim that there either exists a row r of A such that each 0 in row r lies in a column containing another 0, or there exists a column c of A such that each 1 in column c lies in a row containing another 1. Remove this row or column from A to obtain the array A', which is amazing by the lemma above. By construction, every row still has a 1 and every column still has a 0.

To prove the claim, suppose to the contrary that every row of A has a unique 0 in its column, and every column of A has a unique 1 in its row. If A has m rows and n columns, then we have $m \leq n$, since each row has a 0 unique in its column. Similarly, we have $m \geq n$, since each column has a 1 unique in its row. Hence, we have $m = n$, with precisely one 0 in each column and one 1 in each row. But this means that there are n occurrences of 0 and n occurrences of 1 in the entire array, leading to $2n = n^2$ and $n = 2$.

The only 2×2 arrays with precisely one 0 in each column and one 1 in each row are

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}.$$

So A must be one of these possibilities. But these two arrays have the same row and column sums, contradicting the fact that A is amazing. This proves the claim, and so there exists a row r or column c of the desired type.

The claim allows us to successively remove rows or columns from A while preserving its amazingness and so that, at each stage, every row contains a 1 and every column contains a 0. Eventually, we must arrive at a $1 \times n$ or $n \times 1$ or 2×2 array.

If we arrive at a $1 \times n$ array, then as every column contains a 0, so the entire array is 0, contradicting that every row contains a 1. Similarly, if we arrive at an $n \times 1$ array, then as every row contains a 1, so the entire array is 1, contradicting that every column contains a 0. If we arrive at a 2×2 array, the only arrays with a 1 in every row and a 0 in every column are the two arrays shown above, neither of which is amazing.

In any case, we obtain a contradiction. Hence, an amazing array must have a row containing only the number 0 or a column containing only the number 1.

Solution 4 (Daniel Mathews)

We will use the notion of an amazing array, defined in Solution 1. We also use the lemma from Solution 3.

Suppose for the sake of contradiction that A is an amazing array in which every row contains an entry equal to 1 and every column contains an entry equal to 0. Using the lemma, we may delete any duplicate rows or columns, to obtain an amazing array B in which all rows are distinct, all columns are distinct, every row contains a 1, and every column contains a 0.

If B has two rows with equal sums, then it is not amazing. As the rows are distinct, we can swap them to obtain a distinct array with the same row and column sums. Similarly, if B has two columns with equal sums, then it is not amazing. Thus, all row sums of B are distinct, and all column sums of B are distinct.

Let B have m rows and n columns. As each row contains a 1, there are m distinct row sums, each of which is an integer from 1 to n inclusive — hence, $m \geq n$. As each column contains a 0, there are n distinct column sums, each of which is an integer from 0 to $m-1$ inclusive — hence, $n \leq m$. It follows that $m = n$, the row sums are precisely $1, 2, \ldots, n$, and the column sums are precisely $0, 1, \ldots, n-1$.

Thus the sum of all the elements in the array is both $1+2+\cdots+n$ and $0+1+\cdots+n-1$, a contradiction. Hence, an amazing array A must have a row containing only the number 0 or a column containing only the number 1.

AMOC SENIOR CONTEST STATISTICS

DISTRIBUTION OF AWARDS/SCHOOL YEAR

YEAR	NUMBER OF STUDENTS	NUMBER OF AWARDS				
		Prize	High Distinction	Distinction	Credit	Participation
10	31	0	4	5	9	13
11	31	3	3	2	8	15
Other	19	0	1	3	4	11
Total	81	3	8	10	21	39

SCORE DISTRIBUTION/PROBLEM

PROBLEM NUMBER	NUMBER OF STUDENTS/SCORE								MEAN
	0	1	2	3	4	5	6	7	
1	38	2	1	1	3	4	5	27	3.2
2	21	14	5	3	2	2	5	29	3.5
3	35	17	1	4	0	1	14	9	2.3
4	42	16	2	2	1	2	0	16	1.9
5	45	13	9	4	3	1	0	6	1.3

MEAN SCORE/PROBLEM/SCHOOL YEAR

YEAR	NUMBER OF STUDENTS	MEAN					OVERALL MEAN
		Problem					
		1	2	3	4	5	
10	31	3.9	4.4	2.2	1.6	1.1	13.2
11	31	2.9	3.2	2.4	2.5	1.6	12.6
Other	19	2.5	2.6	2.1	1.2	1.0	9.4
All Years	81	3.2	3.5	2.3	1.9	1.3	12.1

AMOC SENIOR CONTEST RESULTS

TOP ENTRIES:

NAME	SCHOOL	YEAR	TOTAL	AWARD
Alexander Gunning	Glen Waverley Secondary College, VIC	11	35	Prize
Jeremy Yip	Trinity Grammar School, VIC	11	35	Prize
Henry Yoo	Perth Modern School, WA	11	35	Prize
Allen Lu	Sydney Grammar School	11	32	High Distinction
Yang Song	James Ruse Agricultural High School, NSW	11	31	High Distinction
Seyoon Ragavan	Knox Grammar School, NSW	10	29	High Distinction
Thomas Baker	Scotch College, VIC	10	29	High Distinction
Kevin Xian	James Ruse Agricultural High School, NSW	10	27	High Distinction
Michael Chen	Scotch College, VIC	11	27	High Distinction
Matthew Cheah	Penleigh and Essendon Grammar School, VIC	9	27	High Distinction
Leo Li	Christ Church Grammar School, WA	10	27	High Distinction
Justin Wu	James Ruse Agricultural High School, NSW	10	23	Distinction
David Steketee	Hale School, WA	11	23	Distinction
Linus Cooper	James Ruse Agricultural High School, NSW	8	22	Distinction
Yong See Foo	Nossal High School, VIC	10	22	Distinction
Jerry Mao	Caulfield Grammar School Wheelers Hill, VIC	8	21	Distinction
Zoe Schwerkolt	Fintona Girls' School, VIC	10	21	Distinction
Alan Guo	Penleigh and Essendon Grammar School, VIC	11	21	Distinction
William Hu	Christ Church Grammar School, WA	8	20	Distinction
James Manton-Hall	Sydney Grammar School, NSW	10	18	Distinction
William Song	Scotch College, VIC	10	18	Distinction