AMOC SENIOR CONTEST

Tuesday, 11 August 2015
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. A number is called *k-addy* if it can be written as the sum of k consecutive positive integers. For example, the number 9 is 2-addy because $9 = 4+5$ and it is also 3-addy because $9 = 2+3+4$.

(a) How many numbers in the set $\{1, 2, 3, \ldots, 2015\}$ are simultaneously 3-addy, 4-addy and 5-addy?

(b) Are there any positive integers that are simultaneously 3-addy, 4-addy, 5-addy and 6-addy?

2. Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 1$ and

$$a_{m+1} = \frac{1a_1 + 2a_2 + 3a_3 + \cdots + ma_m}{a_m} \qquad \text{for } m \geq 1.$$

Determine the largest integer n such that $a_n < 1\,000\,000$.

3. A group of students entered a mathematics competition consisting of five problems. Each student solved at least two problems and no student solved all five problems. For each pair of problems, exactly two students solved them both.

Determine the minimum possible number of students in the group.

4. Let $ABCD$ be a rectangle with $AB > BC$. Let E be the point on the diagonal AC such that BE is perpendicular to AC. Let the circle through A and E whose centre lies on the line AD meet the side CD at F.

Prove that BF bisects the angle AFC.

5. For a real number x, let $\lfloor x \rfloor$ be the largest integer less than or equal to x.

Find all prime numbers p for which there exists an integer a such that

$$\left\lfloor \frac{a}{p} \right\rfloor + \left\lfloor \frac{2a}{p} \right\rfloor + \left\lfloor \frac{3a}{p} \right\rfloor + \cdots + \left\lfloor \frac{pa}{p} \right\rfloor = 100.$$