AMOC SENIOR CONTEST SOLUTIONS

1. A number is called k-*addy* if it can be written as the sum of k consecutive positive integers. For example, the number 9 is 2-addy because $9 = 4 + 5$ and it is also 3-addy because $9 = 2 + 3 + 4$.

   (a) How many numbers in the set $\{1, 2, 3, \ldots, 2015\}$ are simultaneously 3-addy, 4-addy and 5-addy?

   (b) Are there any positive integers that are simultaneously 3-addy, 4-addy, 5-addy and 6-addy?

Solution (Angelo Di Pasquale)

Since $(a-1) + a + (a+1) = 3a$, the 3-addy numbers are precisely those that are divisible by 3 and greater than 3.

Since $(a-2) + (a-1) + a + (a+1) + (a+2) = 5a$, the 5-addy numbers are precisely those that are divisible by 5 and greater than 10.

Since $(a-1) + a + (a+1) + (a+2) = 4a + 2$, the 4-addy numbers are precisely those that are congruent to 2 modulo 4 and greater than 6.

(a) From the observations above, a positive integer is simultaneously 3-addy, 4-addy and 5-addy if and only if it is divisible by 3, divisible by 5, divisible by 2, and not divisible by 4. Such numbers are of the form $30m$, where m is a positive odd integer.

Since $67 \times 30 = 2010$, the number of elements of the given set that are simultaneously 3-addy, 4-addy and 5-addy is $\frac{68}{2} = 34$.

(b) Since $(a-2) + (a-1) + a + (a+1) + (a+2) + (a+3) = 6a + 3$, all 6-addy numbers are necessarily odd.

On the other hand, we have already deduced that all 4-addy numbers are even.

Therefore, there are no numbers that are simultaneously 4-addy and 6-addy.

2. Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 1$ and

$$a_{m+1} = \frac{1a_1 + 2a_2 + 3a_3 + \cdots + ma_m}{a_m} \qquad \text{for } m \geq 1.$$

Determine the largest integer n such that $a_n < 1\,000\,000$.

Solution 1 (Norman Do)

First, we note that all terms of the sequence are positive rational numbers. Below, we rewrite the defining equation for the sequence in both its original form and with the value of m shifted by 1.

$$\begin{aligned} a_m a_{m+1} &= 1a_1 + 2a_2 + 3a_3 + \cdots + ma_m \\ a_{m+1} a_{m+2} &= 1a_1 + 2a_2 + 3a_3 + \cdots + ma_m + (m+1)a_{m+1} \end{aligned}$$

Subtracting the first equation from the second yields

$$a_{m+1}a_{m+2} - a_m a_{m+1} = (m+1)a_{m+1} \qquad \Rightarrow \qquad a_{m+2} - a_m = m + 1,$$

for all $m \geq 1$.

So for $m = 2k + 1$ an odd positive integer,

$$\begin{aligned} a_{2k+1} - a_1 &= (a_{2k+1} - a_{2k-1}) + (a_{2k-1} - a_{2k-3}) + \cdots + (a_3 - a_1) \\ &= 2k + (2k-2) + \cdots + 2 \\ &= 2\,[k + (k-1) + \cdots + 1] \\ &= k(k+1). \end{aligned}$$

Similarly, for $m = 2k$ an even positive integer,

$$\begin{aligned} a_{2k} - a_2 &= (a_{2k} - a_{2k-2}) + (a_{2k-2} - a_{2k-4}) + \cdots + (a_4 - a_2) \\ &= (2k-1) + (2k-3) + \cdots + 3 \\ &= k^2 - 1. \end{aligned}$$

Using $a_1 = 1$ and $a_2 = 1$, we obtain the formula

$$a_m = \begin{cases} \frac{m^2}{4}, & \text{if } m \text{ is even,} \\ \frac{m^2+3}{4}, & \text{if } m \text{ is odd.} \end{cases}$$

If m is odd, then

$$a_{m+1} - a_m = \frac{(m+1)^2}{4} - \frac{m^2+3}{4} = \frac{2m-2}{4},$$

and if m is even, then

$$a_{m+1} - a_m = \frac{(m+1)^2+3}{4} - \frac{m^2}{4} = \frac{2m+4}{4}.$$

In particular, it follows that $a_2 < a_3 < a_4 < \cdots$. Since $a_{2000} = 1\,000\,000$, the largest integer n such that $a_n < 1\,000\,000$ is 1999.

Solution 2 (Angelo Di Pasquale)

We will prove by induction that $a_{2k-1} = k^2 - k + 1$ and $a_{2k} = k^2$ for each positive integer k. The base case $k = 1$ is true since $a_1 = a_2 = 1$. Now assume that the two formulas hold for $k = 1, 2, \ldots, n$. We will show that they also hold for $k = n + 1$.

Consider the following sequence of equalities.

$$\begin{aligned}
\sum_{i=1}^{2n} ia_i &= \sum_{i=1}^{n}(2i-1)\,a_{2i-1} + \sum_{i=1}^{n} 2i\,a_{2i} \\
&= \sum_{i=1}^{n}(2i-1)\,(i^2 - i + 1) + \sum_{i=1}^{n}(2i)\,(i^2) \quad \text{(by the inductive assumption)} \\
&= \sum_{i=1}^{n} 4i^3 - 3i^2 + 3i - 1 \\
&= \sum_{i=1}^{n} 3i^3 + (i-1)^3 \\
&= 3\sum_{i=1}^{n} i^3 + \sum_{i=1}^{n-1} i^3 \\
&= \frac{3n^2(n+1)^2}{4} + \frac{(n-1)^2 n^2}{4} \qquad \left(\text{since } \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}\right) \\
&= n^2(n^2 + n + 1) \\
&= (n^2 + n + 1)a_{2n} \qquad \text{(by the inductive assumption)}
\end{aligned}$$

It follows that

$$a_{2n+1} = \frac{1a_1 + 2a_2 + 3a_3 + \cdots + 2na_{2n}}{a_{2n}} = n^2 + n + 1.$$

Now consider the following sequence of equalities, which uses the facts derived above that state that $\sum_{i=1}^{2n} ia_i = n^2(n^2 + n + 1)$ and $a_{2n+1} = n^2 + n + 1$.

$$\begin{aligned}
\sum_{i=1}^{2n+1} ia_i &= n^2(n^2 + n + 1) + (2n+1)a_{2n+1} \\
&= n^2 a_{2n+1} + (2n+1)a_{2n+1} \\
&= (n+1)^2 a_{2n+1}
\end{aligned}$$

It follows that

$$a_{2n+2} = \frac{1a_1 + 2a_2 + 3a_3 + \cdots + (2n+1)a_{2n+1}}{a_{2n+1}} = (n+1)^2.$$

So we have shown that the two formulas $a_{2k-1} = k^2 - k + 1$ and $a_{2k} = k^2$ hold for $k = 1, 2, \ldots, n+1$. This completes the induction and the rest of the proof follows Solution 1.

3. A group of students entered a mathematics competition consisting of five problems. Each student solved at least two problems and no student solved all five problems. For each pair of problems, exactly two students solved them both.

   Determine the minimum possible number of students in the group.

Solution (Norman Do)

It is possible that the group comprised six students, as demonstrated by the following example.

- Student 1 solved problems 1, 2, 3, 4.
- Student 2 solved problems 1, 2, 3, 5.
- Student 3 solved problems 1, 4, 5.
- Student 4 solved problems 2, 4, 5.
- Student 5 solved problems 3 and 4.
- Student 6 solved problems 3 and 5.

Suppose that a students solved 4 problems, b students solved 3 problems, and c students solved 2 problems. Therefore, a students solved $\binom{4}{2} = 6$ pairs of problems, b students solved $\binom{3}{2} = 3$ pairs of problems and c students solved $\binom{2}{2} = 1$ pair of problems. Since we have shown an example in which the number of students in the group is 6, let us assume that $a + b + c \leq 5$.

There are $\binom{5}{2} = 10$ pairs of problems altogether and, for each pair of problems, exactly two students solved them both, so we must have

$$6a + 3b + c = 20.$$

Reading the above equation modulo 3 yields $c \equiv 2 \pmod 3$. If $c \geq 5$, then we have $a+b+c \geq 6$, contradicting our assumption. Therefore, we must have $c = 2$ and $2a+b = 6$. For $a + b + c \leq 5$, the only solution is given by $(a, b, c) = (3, 0, 2)$.

However, it is impossible for 3 students to have solved 4 problems each. That would mean that each of the 3 students did not solve exactly 1 problem. So there would exist a pair of problems for which 3 students solved them both, contradicting the required conditions.

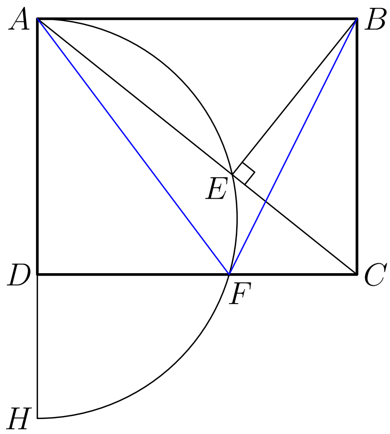
In conclusion, the minimum possible number of students in the group is 6.

4. Let $ABCD$ be a rectangle with $AB > BC$. Let E be the point on the diagonal AC such that BE is perpendicular to AC. Let the circle through A and E whose centre lies on the line AD meet the side CD at F.

   Prove that BF bisects the angle AFC.

Solution 1 (Alan Offer)

Let the circle through A and E whose centre lies on AD meet the line AD again at H.



Since FD is the altitude of the right-angled triangle AFH, we have $\triangle AFH \sim \triangle ADF$. Since triangles AEH and ADC are right-angled with a common angle at A, we have $\triangle AEH \sim \triangle ADC$. Since BE is the altitude of the right-angled triangle ABC, we have $\triangle ABC \sim \triangle AEB$. These three pairs of similar triangles lead respectively to the three pairs of equal ratios

$$\frac{AF}{AD} = \frac{AH}{AF} \qquad \frac{AE}{AD} = \frac{AH}{AC} \qquad \frac{AB}{AE} = \frac{AC}{AB}.$$

Putting these together, we have

$$AF^2 = AD \cdot AH = AC \cdot AE = AB^2.$$

So triangle BAF is isosceles and we have

$$\angle AFB = \angle ABF = 90^\circ - \angle CBF = \angle CFB,$$

where the last equality uses the angle sum in triangle BCF. Since $\angle AFB = \angle CFB$, we have proven that BF bisects the angle AFC.

Solution 2 (Angelo Di Pasquale)

First, note that the circumcircles of triangle AEF and triangle BEC are tangent to the line AB at A and B, respectively. Considering the power of the point A with respect to the circumcircle of triangle BEC, we have

$$AB^2 = AE \cdot AC.$$

Next, by the alternate segment theorem and the fact that $AB \parallel CD$, we have $\angle EFA = \angle EAB = \angle ECF$. Hence, by the alternate segment theorem again, the circumcircle of triangle EFC is tangent to the line AF at F. Considering the power of the point A with respect to the circumcircle of triangle EFC, we have

$$AF^2 = AE \cdot AC.$$

Comparing the two equations above, we deduce that $AB = AF$.

Hence, $\angle AFB = \angle ABF = \angle CFB$, as required.

Solution 3 (Ivan Guo)

Let the circle through A and E whose centre lies on AD meet the line AD again at H. Then $\angle AEB = \angle AEH = 90°$, so the points B, E and H are collinear. Consider the inversion f with centre A and radius AF.

Note that the circumcircle of the cyclic quadrilateral $AEFH$ must map to a line parallel to AB through F. Thus,

$$f(\text{circle } AEFH) = \text{line } CD.$$

This immediately gives $f(C) = E$ and $f(H) = D$. Now the circumcircle of $ABCD$ must map to a line passing through $f(C) = E$ and $f(D) = H$. This implies that $f(B) = B$. Hence, $AB = AF$.

Therefore, we have $\angle BFC = \angle ABF = \angle AFB$.

Solution 4 (Chaitanya Rao)

Consider the following chain of equalities.

$$\begin{aligned}
DF^2 &= AD \cdot DH && (\triangle ADF \sim \triangle FDH)\\
&= AD \cdot (AH - AD) &&\\
&= AD \cdot \left(\frac{AB^2}{BC} - AD\right) && (\triangle ABC \sim \triangle HAB)\\
&= AB^2 - AD^2 && (AD = BC)
\end{aligned}$$

Hence, $AB^2 = DF^2 + AD^2 = AF^2$, which implies that $AB = AF$.

So triangle BAF is isosceles and we have

$$\angle AFB = \angle ABF = 90° - \angle CBF = \angle CFB,$$

where the last equality uses the angle sum in triangle BCF. Since $\angle AFB = \angle CFB$, we have proven that BF bisects the angle AFC.

5. For a real number x, let $\lfloor x \rfloor$ be the largest integer less than or equal to x.

Find all prime numbers p for which there exists an integer a such that

$$\left\lfloor \frac{a}{p} \right\rfloor + \left\lfloor \frac{2a}{p} \right\rfloor + \left\lfloor \frac{3a}{p} \right\rfloor + \cdots + \left\lfloor \frac{pa}{p} \right\rfloor = 100.$$

Solution 1 (Norman Do)

The possible values for p are 2, 5, 17 and 197.

We divide the problem into the following two cases.

- *The number a is divisible by p.*
  If we write $a = kp$, the equation becomes
  $$\frac{a(p+1)}{2} = 100 \qquad \Rightarrow \qquad kp(p+1) = 200.$$
  So both p and $p+1$ are positive divisors of 200. However, one can easily see that there are no such primes p.
- *The number a is not divisible by p.*
  For a real number x, let $\{x\} = x - \lfloor x \rfloor$. Then we may write the equation as
  $$\left(\frac{a}{p} + \frac{2a}{p} + \frac{3a}{p} + \cdots + \frac{pa}{p}\right) - \left(\left\{\frac{a}{p}\right\} + \left\{\frac{2a}{p}\right\} + \left\{\frac{3a}{p}\right\} + \cdots + \left\{\frac{pa}{p}\right\}\right) = 100.$$
  Summing the terms of the arithmetic progression on the left-hand side yields
  $$\frac{a(p+1)}{2} - \left[\left\{\frac{a}{p}\right\} + \left\{\frac{2a}{p}\right\} + \left\{\frac{3a}{p}\right\} + \cdots + \left\{\frac{pa}{p}\right\}\right] = 100.$$
  We will prove that the sequence of numbers $\left\{\frac{a}{p}\right\}, \left\{\frac{2a}{p}\right\}, \left\{\frac{3a}{p}\right\}, \ldots, \left\{\frac{pa}{p}\right\}$ is a rearrangement of the sequence of numbers $\frac{0}{p}, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$.
  Observe that if k is a positive integer, then $\left\{\frac{ka}{p}\right\}$ is one of the numbers $\frac{0}{p}, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$. So it suffices to show that no two of the numbers $\left\{\frac{a}{p}\right\}, \left\{\frac{2a}{p}\right\}, \left\{\frac{3a}{p}\right\}, \ldots, \left\{\frac{pa}{p}\right\}$ are equal. Suppose for the sake of contradiction that $\left\{\frac{ia}{p}\right\} = \left\{\frac{ja}{p}\right\}$, where $1 \le i < j \le p$. Then $\frac{ja}{p} - \frac{ia}{p} = \frac{a(j-i)}{p}$ must be an integer. It follows that either a is divisible by p or $j - i$ is divisible by p. However, since we have assumed that a is not divisible by p and that $1 \le i < j \le p$, we obtain the desired contradiction. Hence, we may conclude that the sequence of numbers $\left\{\frac{a}{p}\right\}, \left\{\frac{2a}{p}\right\}, \left\{\frac{3a}{p}\right\}, \ldots, \left\{\frac{pa}{p}\right\}$ is a rearrangement of the sequence of numbers $\frac{0}{p}, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$.
  We may now write the equation as
  $$\frac{a(p+1)}{2} - \frac{p-1}{2} = 100 \qquad \Rightarrow \qquad (a-1)(p+1) = 198.$$
  Therefore, $p+1$ is a positive divisor of 198 — in other words, one of the numbers
  $$1, 2, 3, 6, 9, 11, 18, 22, 33, 66, 99, 198.$$
  Since p is a prime, it follows that p must be equal to 2, 5, 17 or 197. This leads to the possible solutions $(p, a) = (2, 67), (5, 34), (17, 12), (197, 2)$. All four of these pairs satisfy the given equation with a not divisible by p, so we obtain $p = 2, 5, 17, 197$.

Solution 2 (Ivan Guo, Angelo Di Pasquale and Ian Wanless)

The possible values for p are 2, 5, 17 and 197.

The case where a is divisible by p is handled in the same way as Solution 1.

Furthermore, one can check that the pair $(p, a) = (2, 67)$ satisfies the conditions of the problem. So assume that p is an odd prime and that a is not divisible by p.

For any integers $1 \leq r, s \leq p-1$ with $r+s=p$, we have $p \nmid ra$ and $p \nmid sa$. Therefore,

$$\frac{ar}{p} - 1 + \frac{as}{p} - 1 < \left\lfloor \frac{ar}{p} \right\rfloor + \left\lfloor \frac{as}{p} \right\rfloor < \frac{ar}{p} + \frac{as}{p} \quad \Rightarrow \quad a - 2 < \left\lfloor \frac{ar}{p} \right\rfloor + \left\lfloor \frac{as}{p} \right\rfloor < a.$$

But since $\left\lfloor \frac{ar}{p} \right\rfloor + \left\lfloor \frac{as}{p} \right\rfloor$ is an integer, we conclude that

$$\left\lfloor \frac{ar}{p} \right\rfloor + \left\lfloor \frac{as}{p} \right\rfloor = a - 1.$$

The numbers $\{1, 2, \ldots, p-1\}$ can be partitioned into $\frac{p-1}{2}$ pairs whose sum is p.

Using the equation above for each such pair and substituting into the original equation, we obtain

$$\left(\frac{p-1}{2}\right)(a-1) + a = 100 \qquad \Rightarrow \qquad (a-1)(p+1) = 198.$$

Therefore, $p+1$ is a positive divisor of 198 — in other words, one of the numbers

$$1, 2, 3, 6, 9, 11, 18, 22, 33, 66, 99, 198.$$

Since p is a prime, it follows that p must be equal to 2, 5, 17 or 197. This leads to the possible solutions $(p, a) = (2, 67), (5, 34), (17, 12), (197, 2)$. All four of these pairs satisfy the given equation with a not divisible by p, so we obtain $p = 2, 5, 17, 197$.

AMOC SENIOR CONTEST RESULTS

Name	School	Year	Score
Prize			
Yong See Foo	Nossal High School VIC	11	35
Kevin Xian	James Ruse Agricultural High School NSW	11	35
Wilson Zhipu Zhao	Killara High School NSW	11	35
Ilia Kucherov	Westall Secondary College VIC	11	34
Seyoon Ragavan	Knox Grammar School NSW	11	34
High Distinction			
Jongmin Lim	Killara High School NSW	11	32
Matthew Cheah	Penleigh and Essendon Grammar School VIC	10	29
Jerry Mao	Caulfield Grammar School (Wheelers Hill) VIC	9	29
Distinction			
Alexander Barber	Scotch College VIC	11	28
Michelle Chen	Methodist Ladies’ College VIC	11	28
Thomas Baker	Scotch College VIC	11	27
Linus Cooper	James Ruse Agricultural High School NSW	9	27
Steven Lim	Hurlstone Agricultural High School NSW	9	27
Eric Sheng	Newington College NSW	11	27
William Song	Scotch College VIC	11	27
Jack Liu	Brighton Grammar VIC	9	26
Leo Li	Christ Church Grammar School WA	11	25
Bobby Dey	James Ruse Agricultural High School NSW	10	24
Michael Robertson	Dickson College ACT	11	24
Charles Li	Camberwell Grammar School VIC	9	22
Isabel Longbottom	Rossmoyne Senior High School WA	10	22
Guowen Zhang	St Joseph’s College, Gregory Terrace QLD	9	22

AMOC SENIOR CONTEST STATISTICS

SCORE DISTRIBUTION/PROBLEM

Problem Number	Number of Students/Score								Mean
	0	1	2	3	4	5	6	7	
1	1	0	0	0	1	5	46	31	6.2
2	16	9	8	1	2	8	15	25	4.1
3	25	4	11	2	4	5	8	25	3.5
4	54	13	2	0	0	0	0	15	1.5
5	65	2	1	0	0	4	6	6	1.2