AMOC SENIOR CONTEST SOLUTIONS

1. Determine all triples (a, b, c) of distinct integers such that a, b, c are solutions of

$$x^3 + ax^2 + bx + c = 0.$$

Solution 1 (Norman Do)

Using Vieta's formulas to relate the coefficients of the polynomial to its roots, we obtain the following equations.

$$\begin{aligned} a + b + c &= -a \\ ab + bc + ca &= b \\ abc &= -c \end{aligned}$$

The third equation implies that one of the following two cases must hold.

- Case 1: $c = 0$
  The equations then reduce to $a+b = -a$ and $ab = b$. The second of these implies that either $b = 0$ or $a = 1$. Since a, b, c are distinct, we cannot have $b = 0$. So $a = 1$ and the first equation implies that $b = -2$, which leads to the triple $(a, b, c) = (1, -2, 0)$.
- Case 2: $ab = -1$
  Since a and b are integers, we must have $(a, b) = (1, -1)$ or $(a, b) = (-1, 1)$. Using the first equation above, this yields the triples $(a, b, c) = (1, -1, -1)$ and $(a, b, c) = (-1, 1, 1)$. Since we require a, b, c to be distinct, we do not obtain a solution in this case.

We can now check that the polynomial $x^3 + x^2 - 2x = x(x - 1)(x + 2)$ does indeed have the roots $1, -2, 0$. So the only triple that satisfies the conditions of the problem is $(a, b, c) = (1, -2, 0)$.

Solution 2 (Angelo Di Pasquale and Alan Offer)

Substitute $x = a, b, c$ into the polynomial to find

$$2a^3 + ab + c = 0 \tag{1}$$

$$b^3 + ab^2 + b^2 + c = 0 \tag{2}$$

$$c^3 + ac^2 + bc + c = 0. \tag{3}$$

- Case 1: At least one of a, b or c is equal to zero.
  In this case, the above equations imply that $c = 0$.

Since a, b, c are distinct, we have $a, b \neq 0$. Thus, equations (1) and (2) simplify to

$$2a^2 + b = 0 \qquad \text{and} \qquad b + a + 1 = 0.$$

Eliminating b from these yields the equation

$$2a^2 - a - 1 = 0 \qquad \Rightarrow \qquad (a-1)(2a+1) = 0.$$

Since a is an integer, we have $a = 1$. It follows from equation (1) that $b = -2$.

- Case 2: All of a, b and c are not equal to zero.
From equation (2), we have $b \mid c$. From equation (3), we have $c^2 \mid c(b+1)$, so $c \mid b+1$. Thus, $b \mid b+1$, and so $b \mid 1$.
If $b = -1$, then equations (1) and (2) become

$$2a^3 - a + c = 0 \qquad \text{and} \qquad a + c = 0.$$

Eliminating c from these yields $a^3 = a$. But since a, b, c are distinct and non-zero, we have $a = 1$. Then equation (1) yields $c = -1$ and $b = -1$, a contradiction.
If $b = 1$, then equations (1) and (2) become

$$2a^3 + a + c = 0 \qquad \text{and} \qquad a + c + 2 = 0.$$

Eliminating c from these yields $a^3 = 1$. Thus $a = 1$ and $b = 1$, a contradiction.

It is routine to verify that $(a, b, c) = (1, -2, 0)$ solves the problem. So the only triple that satisfies the conditions of the problem is $(a, b, c) = (1, -2, 0)$.

2. Each unit square in a 2016×2016 grid contains a positive integer. You play a game on the grid in which the following two types of moves are allowed.

- Choose a row and multiply every number in the row by 2.
- Choose a positive integer, choose a column, and subtract the positive integer from every number in the column.

You win if all of the numbers in the grid are 0. Is it always possible to win after a finite number of moves?

Solution (Norman Do)

We will prove that it is possible to make every number in an $m \times n$ grid equal to 0 after a finite number of moves. The proof will be by induction on n, the number of columns in the grid.

Consider the base case, in which the number of columns is 1. Let the difference between the maximum and minimum numbers in the column be D. We will show that if $D > 0$, then it is possible to reduce the value of D after a finite number of moves. First, we subtract a positive integer from every number in the column to make the minimum number in the column 1. Now multiply all of the entries equal to 1 by 2, which reduces D by 1. Therefore, after a finite number of moves, it is possible to make all numbers in the column equal to each other. By subtracting this number from each entry of the column, we have made every number in the column equal to 0.

Now consider an $m \times n$ grid with $n \geq 2$. Suppose that we can make every number in a grid with $n - 1$ columns equal to 0 after a finite number of moves. We simply apply the construction of the previous paragraph to the leftmost column in the grid. Note that the entries in the remaining columns may change, but remain positive. Therefore, after a finite number of moves, we obtain a grid whose leftmost column contains only 0. By the inductive hypothesis, we can make every number in the remaining $m \times (n - 1)$ grid equal to 0 after a finite number of moves. Furthermore, observe that these moves do not change the 0 entries in the leftmost column. Therefore, it is always possible to make every number in an $m \times n$ grid equal to 0 after a finite number of moves.

3. Show that in any sequence of six consecutive integers, there is at least one integer x such that

$$(x^2+1)(x^4+1)(x^6-1)$$

is a multiple of 2016.

Solution 1 (Alan Offer)

Let $N = (x^2+1)(x^4+1)(x^6-1)$. Suppose that x is relatively prime to 2016, whose prime factorisation is $2^5 \times 3^2 \times 7^1$.

- Since x is not divisible by 7, Fermat's little theorem tells us that $x^6 - 1$ is divisible by 7. Therefore, $7 \mid N$.
- Since x is relatively prime to 9, Euler's theorem tells us that $x^{\varphi(9)} - 1 = x^6 - 1$ is divisible by 9. Therefore, $9 \mid N$.
- Since x is odd, the expressions x^2+1, x^4+1, $x+1$ and $x-1$ are all even, with one of the last two necessarily a multiple of 4. Therefore,

  $$N = (x^2+1)(x^4+1)(x+1)(x-1)(x^4+x^2+1)$$

  is divisible by 2^5.

Thus, if x is relatively prime to 2016, then N is a multiple of 2016.

Now consider the following residues modulo 42: 1, 5, 11, 17, 23, 25, 31, 37, 1. Note that the largest difference between consecutive numbers in this sequence is 6. It follows that, among any six consecutive integers, there is an integer x that is congruent modulo 42 to one of these residues. Since these residues are relatively prime to 42, it follows that x is relatively prime to 42. And since 42 and 2016 have the same prime factors — namely 2, 3 and 7 — it follows that x is relatively prime to 2016. So, by the above reasoning, N is a multiple of 2016.

Solution 2 (Angelo Di Pasquale, Chaitanya Rao and Jamie Simpson)

Let $N = (x^2+1)(x^4+1)(x^6-1)$. Among any six consecutive integers, there must be one of the form $6k+1$ and one of the form $6k-1$ for some integer k. At most one of these numbers can be divisible by 7, so let x be one that is not divisible by 7.

- Observe that

  $$N = (x^2+1)(x^4+1)(x-1)(x^2+x+1)(x+1)(x^2-x+1).$$

  One may check directly that if x is congruent to 1, 2, 3, 4, 5, or 6 modulo 7, then, the third, fourth, sixth, fourth, sixth, or fifth factor is divisible by 7, respectively. In any case, N is divisible by 7.
- If $x = 6k+1$, then the third and fourth factors in the expression above are divisible by 3. If $x = 6k-1$, then the fifth and sixth factors are divisible by 3. In any case, N is divisible by 9.

- Since x is odd, we have $x^2 \equiv 1 \pmod 8$, so $x^6 \equiv 1 \pmod 8$. Furthermore, x^2+1 and x^4+1 are even. So $N = (x^2+1)(x^4+1)(x^6-1)$ is divisible by $2^3 \times 2 \times 2 = 32$.

We conclude that in any sequence of six consecutive integers, there is at least one integer x such that $N = (x^2+1)(x^4+1)(x^6-1)$ is divisible by 7, 9, and 32. Hence, N is divisible by their lowest common multiple $7 \times 9 \times 32 = 2016$.

Solution 3 (Kevin McAvaney)

Note that

$$\begin{aligned}(x^2+1)(x^4+1)(x^6-1) &= (x^2+1)(x^4+1)(x^2-1)(x^4+x^2+1)\\ &= (x^8-1)(x^4+x^2+1).\end{aligned}$$

- Let $x = 7k + r$, where $r = 0, 1, 2, 3, 4, 5$ or 6. From the binomial theorem, we know that $x^6 = (\text{a multiple of } 7) + r^6$. By testing each of the values for r, we see that $7 \mid x^6 - 1$ if r is not equal to 0.
- Let $x = 3k + r$, where $r = 0, 1$ or 2. From the binomial theorem, we know that $x^6 = (\text{a multiple of } 9) + r^6$. By testing each of the values of r, we see that $9 \mid x^6 - 1$ if r is not equal to 0.
- Let $x = 2k + 1$. From the binomial theorem, we know that

  $$x^8 = (\text{a multiple of } 32) + 28(2k)^2 + 8(2k) + 1 = (\text{a multiple of } 32) + 16k(7k+1) + 1.$$

  Hence, $32 \mid x^8 - 1$.

In any six consecutive integers, exactly three are odd and they take the form $a, a+2, a+4$ for some integer a. Of these, exactly one is a multiple of 3 and at most one is a multiple of 7. So there is at least one of them, x say, that is not divisible by 2, 3 or 7. By the above reasoning, it follows that this value of x makes $(x^2+1)(x^4+1)(x^6-1)$ divisible by 2016.

4. Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 1$ and

$$a_n = n - \lfloor \sqrt{a_{n-1}} \rfloor, \qquad \text{for } n \geq 2.$$

Determine the value of a_{800}.

(Here, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.)

Solution 1 (Norman Do)

The first several terms of the sequence are $1, 1, 2, 3, 4, 4, 5, 6, 7, 8, 9, 9, 10, \ldots$. It appears that the following are true.

- We have $a_n - a_{n-1} = 0$ or 1 for all $n \geq 2$, so the sequence is non-decreasing and contains every positive integer.
- Every positive integer occurs exactly once, apart from perfect squares, which appear exactly twice.

First, we prove by induction that $a_n - a_{n-1} = 0$ or 1 for all $n \geq 2$. Note that this is certainly true for the first several terms listed above. Using the inductive hypothesis that $a_n - a_{n-1} = 0$ or 1, we have

$$\begin{aligned} a_{n+1} - a_n &= (n + 1 - \lfloor \sqrt{a_n} \rfloor) - (n - \lfloor \sqrt{a_{n-1}} \rfloor) \\ &= 1 + \lfloor \sqrt{a_{n-1}} \rfloor - \lfloor \sqrt{a_n} \rfloor \\ &= \begin{cases} 1, & \text{if } a_n = a_{n-1}, \\ 1 + \lfloor \sqrt{a_n - 1} \rfloor - \lfloor \sqrt{a_n} \rfloor, & \text{if } a_n = a_{n-1} + 1. \end{cases} \end{aligned} \tag{1}$$

For all positive integers a, we have

$$\left[\sqrt{a} - \sqrt{a-1}\right]^2 = 2a - 1 - 2\sqrt{a}\sqrt{a-1} < 2a - 1 - 2\sqrt{a-1}\sqrt{a-1} = 1.$$

So we have $\sqrt{a} - \sqrt{a-1} < 1$, which implies that $0 \leq \lfloor \sqrt{a} \rfloor - \lfloor \sqrt{a-1} \rfloor \leq 1$. Combining this with equation (1), we obtain $a_{n+1} - a_n = 0$ or 1. Therefore, by induction, we have $a_n - a_{n-1} = 0$ or 1 for all $n \geq 2$, so the sequence is non-decreasing and contains every positive integer.

Next, we prove that every positive integer occurs exactly once, apart from perfect squares, which appear exactly twice. Since we can deduce from equation (1) that

$$a_n - a_{n-1} = 0 \qquad \Rightarrow \qquad a_{n+1} - a_n = 1,$$

every positive integer occurs either once or twice. From equation (1) again, we know that a term occurs twice when $a_{n+1} - a_n = 0$, which arises only if $\lfloor \sqrt{a_n} \rfloor - \lfloor \sqrt{a_n - 1} \rfloor = 1$ or $n = 1$. However, this occurs only if a_n is a perfect square.

Finally, we observe that the sequence can be defined from the following properties.

- We have $a_n - a_{n-1} = 0$ or 1 for all $n \geq 2$, so the sequence is non-decreasing and contains every positive integer.

- Every positive integer occurs exactly once, apart from perfect squares, which appear exactly twice.

It follows that the only time that the number $27^2 - 1 = 728$ appears in the sequence is at term number $728 + 26 = 754$. Then $a_{755} = 729, a_{756} = 729, a_{757} = 730, a_{758} = 731, \ldots$ all the way up to $a_{800} = 773$. In this part of the sequence, no term appears twice, since there is no square number between 731 and 773.

Solution 2 (Angelo Di Pasquale and Alan Offer)

The function $f(m) = m^2 + m$ is strictly increasing on the non-negative integers. Hence, there is a unique non-negative integer m such that $f(m) \le n < f(m+1)$ for each positive integer n. It follows that any positive integer n may be written uniquely in the form

$$n = f(m) + k, \tag{1}$$

where m is a non-negative integer and k is an integer with $0 \le k \le 2m+1$. We shall refer to the form in (1) as the *special* form of n.

We shall prove by induction that if n has special form $n = f(m) + k$, then $a_n = m^2 + k$. For the base case $n = 1$, we have $1 = f(0) + 1$, and $a_1 = 0^2 + 1 = 1$, as desired. Assume that $a_n = m^2 + k$, where n has special form $n = f(m) + k$. We compute a_{n+1}.

- Case 1: $k \le 2m$
  Since $k + 1 \le 2m + 1$, the special form of $n + 1$ is given by $n + 1 = f(m) + k + 1$. Also observe that $\lfloor\sqrt{a_n}\rfloor = m$. Hence,

  $$a_{n+1} = n + 1 - \lfloor\sqrt{a_n}\rfloor = f(m) + k + 1 - m = m^2 + k + 1,$$

  as desired.
- Case 2: $k = 2m + 1$
  Note that $n+1 = f(m)+k+1 = m^2+m+2m+1+1 = (m+1)^2+(m+1)$. Thus the special form of $n+1$ is given by $n+1 = f(m+1)+0$. Observe that $\lfloor\sqrt{a_n}\rfloor = m+1$. Hence,

  $$a_{n+1} = n + 1 - \lfloor\sqrt{a_n}\rfloor = f(m + 1) - (m + 1) = (m + 1)^2 + 0,$$

  as desired. This concludes the induction.

To conclude, note that $f(27) = 756$ and $f(28) = 812$. Hence, 800 has special form $800 = f(27) + 44$. It follows that $a_{800} = 27^2 + 44 = 773$.

Solution 3 (Daniel Mathews)

Let $a_n = n - b_n$. For any $n \ge 1$, we claim that $b_n = k$ for $k(k+1) \le n \le (k+1)(k+2) - 1$.

We note that, for each $n \ge 1$, there is precisely one integer $k \ge 0$ such that the above inequality holds, so the above gives a well-defined formula for b_n. We observe that the claimed value for b_n is true for $n = 1$; now assume it is true for all b_n, and we prove it is true for b_{n+1}.

We have $a_{n+1} = n + 1 - \lfloor\sqrt{a_n}\rfloor$, and $a_n = n - k$, where $k(k + 1) \le n \le (k + 1)(k + 2) - 1$ as above.

Thus $k^2 \le n - k \le k^2 + 2k + 1 = (k + 1)^2$, and hence, $k \le \sqrt{a_n} \le k + 1$. Thus $\lfloor\sqrt{a_n}\rfloor = k$ or $k+1$, and $\lfloor\sqrt{a_n}\rfloor = k+1$ if and only if $n = (k+1)(k+2)-1$. Hence, we can consider the two cases separately: $n = (k+1)(k+2)-1$, and $k(k+1) \le n \le (k+1)(k+2)-2$. In the first case $\lfloor\sqrt{a_n}\rfloor = k + 1$, and in the second case $\lfloor\sqrt{a_n}\rfloor = k$.

So, first suppose $n = (k + 1)(k + 2) - 1$ and $\lfloor\sqrt{a_n}\rfloor = k + 1$. Then $a_{n+1} = n + 1 - (k + 1)$, so $b_{n+1} = k + 1$. And indeed, since $(k + 1)(k + 2) - 1 = n$, we have the inequality $(k + 1)(k + 2) \le n + 1 \le (k + 1)(k + 2) - 1$, so the formula for b_n holds.

Now suppose that $k(k+1) \le n \le (k+1)(k+2)-2$ and $\lfloor\sqrt{a_n}\rfloor = k$. Then $a_{n+1} = n+1-k$, and $b_{n+1} = k$.

And since $k(k + 1) \le n \le (k + 1)(k + 2) - 2 < (k + 1)(k + 2) - 1$ the formula holds for b_n in this case also.

By induction, then our formula holds for all n. When $n = 800$ we have $k = 27$, so $b_{800} = 27$ and $a_{800} = 773$.

Solution 4 (Kevin McAvaney)

The first few terms of the sequence are $1, 1, 2, 3, 4, 4, 5, 6, 7, 8, 9, 9, 10, \ldots$. We want to show this pattern continues (terms increase by 1, except each square appears exactly twice).

Suppose for some k and m, $a_{k-1} = a_k = m^2$. Then $a_k = k-[\sqrt{m^2}] = k-m$, so $m^2 = k-m$. Hence, $a_{k+1} = (k + 1) - [\sqrt{m^2}] = (k + 1) - m = m^2 + 1$.

For $1 \le r \le (m + 1)^2 - m^2$, we have inductively $a_{k+r} = (k + r) - [\sqrt{m^2 + r - 1}] = (k + r) - m = m^2 + r$. In particular, $a_{k+(m+1)^2-m^2} = m^2 + (m + 1)^2 - m^2 = (m + 1)^2$. Hence $a_{k+(m+1)^2-m^2+1} = (k+2m+2)-(m+1) = k+(m+1) = m^2+m+(m+1) = (m+1)^2$. So the pattern continues.

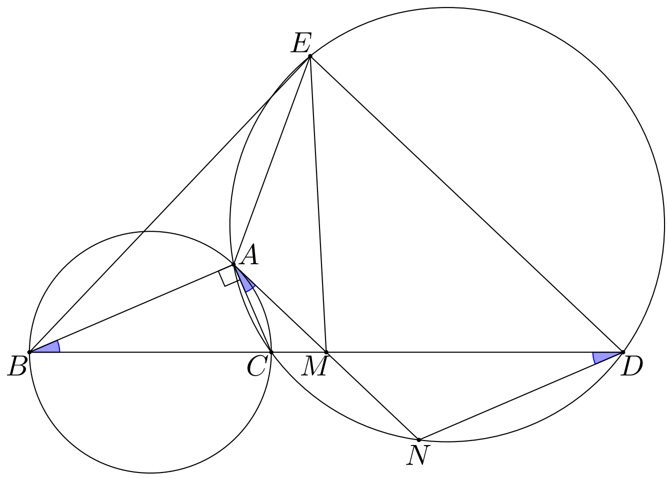
Moreover, the index for the second appearance of the square m^2 is $k = m^2 + m$. The largest m such that $m^2 + m \le 800$ is 27 and $27^2 + 27 = 756$. Since $a_{756} = 27^2 = 729$ and $800 - 756 = 44$, we have $a_{800} = 729 + 44 = 773$.

5. Triangle ABC is right-angled at A and satisfies $AB > AC$. The line tangent to the circumcircle of triangle ABC at A intersects the line BC at M. Let D be the point such that M is the midpoint of BD. The line through D that is parallel to AM intersects the circumcircle of triangle ACD again at E.

   Prove that A is the incentre of triangle EBM.

Solution 1 (Angelo Di Pasquale)

Let the line AM intersect the circumcircle of triangle ACD again at N.



Using the alternate segment theorem in circle ABC, and then the cyclic quadrilateral $ACND$, we have

$$\angle CBA = \angle CAN = \angle CDN.$$

Since $MB = MD$, it follows that $\triangle MAB \cong \triangle MND$ (ASA). Hence, $MA = MN$.

We also have $\triangle MND \cong \triangle MAE$, since these triangles are related by reflection in the perpendicular bisector of the parallel chords AN and ED in circle $ANDE$. Hence, $\triangle MAB \cong \triangle MAE$ and MA is a line of symmetry for triangle EBM.

Now let $x = \angle AEM = \angle MBA = \angle CAM$ and $y = \angle ABE = \angle BEA$. The angle sum in $\triangle ABM$ yields $\angle AMB = 90^\circ - 2x$. By symmetry, we also have $\angle EMA = 90^\circ - 2x$.

Finally, the angle sum in $\triangle EBM$ yields $2x + 2y + 2(90^\circ - 2x) = 180^\circ$, which implies that $x = y$.

Thus, A is the incentre of $\triangle EBM$ because it is the intersection of its angle bisectors.

Solution 2 (Angelo Di Pasquale)

With N defined as in the solution above, we have

$$\begin{aligned} MA \cdot MN &= MC \cdot MD && \text{(power of } M \text{ with respect to circle } ACD) \\ &= MC \cdot MB && (M \text{ is the midpoint of } BD) \\ &= MA^2. && \text{(power of } M \text{ with respect to circle } ABC) \end{aligned}$$

So $MA = MN$ and since $MB = MD$, we have $\triangle MAB \cong \triangle MND$ (SAS). Hence, $MA = MN$.

The rest is as in the solution above.

AMOC SENIOR CONTEST RESULTS

Name	School	Year	Score
Gold			
Linus Cooper	James Ruse Agricultural High School NSW	Year 10	35
William Hu	Christ Church Grammar School WA	Year 10	35
Jack Liu	Brighton Grammar School VIC	Year 10	35
Hadyn Tang	Trinity Grammar School VIC	Year 7	35
Guowen Zhang	St Joseph's College (Gregory Terrace) QLD	Year 10	35
Matthew Cheah	Penleigh and Essendon Grammar School VIC	Year 11	31
Bobby Dey	James Ruse Agricultural High School NSW	Year 11	29
Sharvil Kesarwani	Merewether High School NSW	Year 9	29
Jerry Mao	Caulfield Grammar School VIC	Year 10	29
Silver			
James Bang	Baulkham Hills High School NSW	Year 9	28
Charles Li	Camberwell Grammar School VIC	Year 10	28
Jeff (Zefeng) Li	Camberwell Grammar School VIC	Year 9	28
Isabel Longbottom	Rossmoyne Senior High School WA	Year 11	28
Stanley Zhu	Melbourne Grammar School VIC	Year 10	27
Anthony Tew	Pembroke School SA	Year 10	25
Yuelin Shen	Scotch College (Perth) WA	Year 11	24
Kieran Hamley	All Saints Anglican School QLD	Year 11	23
Elliott Murphy	Canberra Grammar School ACT	Year 11	23
Daniel Jones	All Saints Anglican School QLD	Year 11	22
Yasiru Jayasooriya	James Ruse Agricultural High School NSW	Year 8	21
Anthony Pisani	St Paul’s Anglican Grammar School VIC	Year 9	21
Tommy Wei	Scotch College (Melbourne) VIC	Year 10	21
Bronze			
Shivasankaran Jayabalan	Rossmoyne Senior High School WA	Year 10	19
Tony Jiang	Scotch College (Melbourne) VIC	Year 11	19
Steven Lim	Hurlstone Agricultural High School NSW	Year 10	19
Dibyendu Roy	Sydney Boys High School NSW	Year 11	19
Jordan Ka Truong	Sydney Technical High School NSW	Year 11	19

Name	School	Year	Score
Jodie Lee	Seymour College SA	Year 11	18
Phillip Liang	James Ruse Agricultural High School NSW	Year 10	18
Forbes Mailler	Canberra Grammar School ACT	Year 10	16
Hilton Nguyen	Sydney Technical High School NSW	Year 10	16
Ken Gene Quah	East Doncaster Secondary College VIC	Year 8	16
Kieran Shivakumaarun	Sydney Boys High School NSW	Year 11	16
Ruiqian Tong	Presbyterian Ladies' College VIC	Year 10	15
Phillip Huynh	Brisbane State High School QLD	Year 11	14
Oliver Papillo	Camberwell Grammar School VIC	Year 9	14
Ziqi Yuan	Lyneham High School ACT	Year 9	14
HIGH DISTINCTION			
Sharwel Lei	All Saints Anglican School QLD	Year 11	13
Xinyue Alice Zhang	A B Paterson College QLD	Year 10	13
Adrian Lo	Newington College NSW	Year 8	11
Ranit Bose	Lyneham High School ACT	Year 10	10
Ryu Callaway	Lyneham High School ACT	Year 10	10
Liam Coy	Sydney Grammar School NSW	Year 8	10
Anezka Hamdani	Perth Modern School WA	Year 11	9
Arun Jha	Perth Modern School WA	Year 11	9

AMOC SENIOR CONTEST STATISTICS

Mark	Q1	Q2	Q3	Q4	Q5
0	24	44	25	6	52
1	1	0	4	13	8
2	0	5	3	27	0
3	3	0	2	5	0
4	8	2	6	8	0
5	3	5	3	3	1
6	4	6	1	4	1
7	33	14	32	10	14
Total	312	177	285	223	117
Mean	4.1	2.3	3.8	2.9	1.5
Standard Deviation	3.1	3.0	3.1	2.2	2.8