2017 AMOC SENIOR CONTEST

Tuesday, 8 August 2017
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. For each pair of real numbers (r, s), prove that there exists a real number x that satisfies at least one of the following two equations.

$$\begin{aligned} x^2 + (r+1)x + s &= 0 \\ rx^2 + 2sx + s &= 0 \end{aligned}$$

2. Let $ABCD$ be a quadrilateral with AB not parallel to CD. The circle with diameter AB is tangent to the side CD at X. The circle with diameter CD is tangent to the side AB at Y.

Prove that the quadrilateral $BCXY$ is cyclic.

3. Let $a_1 < a_2 < \cdots < a_{2017}$ and $b_1 < b_2 < \cdots < b_{2017}$ be positive integers such that

$$\left(2^{a_1}+1\right)\left(2^{a_2}+1\right)\cdots\left(2^{a_{2017}}+1\right) = \left(2^{b_1}+1\right)\left(2^{b_2}+1\right)\cdots\left(2^{b_{2017}}+1\right).$$

Prove that $a_i = b_i$ for $i = 1, 2, \ldots, 2017$.

4. Find all positive integers $n \geq 5$ for which we can place a real number at each vertex of a regular n-sided polygon, such that the following two conditions are satisfied.

- None of the n numbers is equal to 1.
- For each vertex of the polygon, the sum of the numbers at the nearest four vertices is equal to 4.

5. Let n be a positive integer. Consider $2n$ points equally spaced around a circle. Suppose that n of the points are coloured blue and the remaining n points are coloured red. We write down the distance between each pair of blue points in a list, from shortest to longest. We write down the distance between each pair of red points in another list, from shortest to longest. (Note that the same distance may occur more than once in a list.)

Prove that the two lists of distances are the same.