AMOC SENIOR CONTEST SOLUTIONS

1. For each pair of real numbers (r, s), prove that there exists a real number x that satisfies at least one of the following two equations.

$$x^2 + (r+1)x + s = 0$$
$$rx^2 + 2sx + s = 0$$

Solution 1 (Norman Do)

In order to obtain a contradiction, suppose that there does not exist a real number x that satisfies at least one of the two equations. The discriminants of the two quadratic equations are $(r+1)^2 - 4s$ and $4s^2 - 4rs$, respectively. Therefore, we have

$$(r+1)^2 - 4s < 0 \qquad \text{and} \qquad 4s^2 - 4rs < 0.$$

Adding these two inequalities, we obtain

$$(r+1)^2 - 4s + 4s^2 - 4rs < 0 \qquad \Rightarrow \qquad (r+1-2s)^2 < 0.$$

Since the square of a real number cannot be negative, this yields a contradiction. It follows that there must exist a real number x that satisfies at least one of the two equations.

Solution 2 (Alice Devillers, Angelo Di Pasquale, Ivan Guo, Dan Mathews, Chaitanya Rao and Ian Wanless)

If $s \le 0$, then the discriminant $(r+1)^2 - 4s$ of the first quadratic equation is a sum of two non-negative numbers. Hence, it is non-negative and the first equation has a real solution.

If $s > 0$ and $s \ge r$, then the discriminant $4s^2 - 4rs$ of the second quadratic equation is non-negative. Hence, the second equation has a real solution.

The only case left to consider is $0 < s < r$. Then the discriminant of the first quadratic equation is

$$(r+1)^2 - 4s > (s+1)^2 - 4s = (s-1)^2 \ge 0.$$

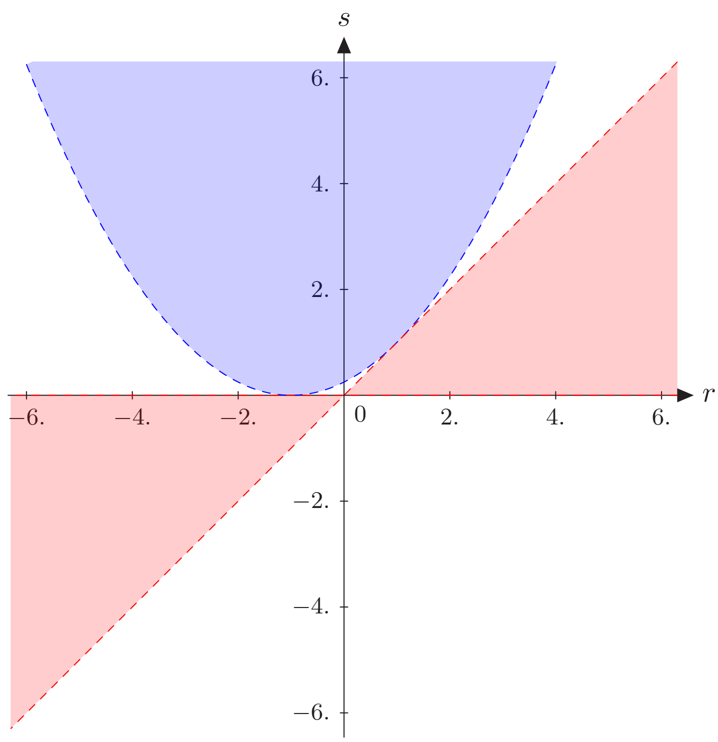
Hence, the first equation has a real solution.

Solution 3 (Angelo Di Pasquale)

As in Solution 1, we wish to show that there do not exist real numbers r and s such that

$$(r+1)^2 - 4s < 0 \qquad \text{and} \qquad 4s^2 - 4rs < 0.$$

One can simply observe this from the graphs of these two inequalities, which are shown in the figure below.



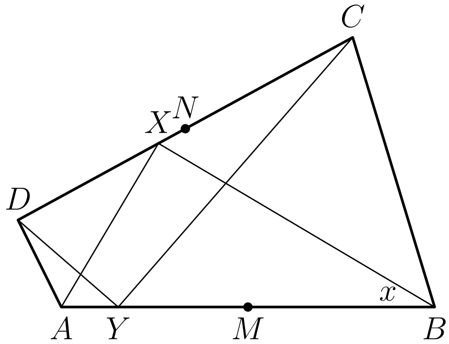
The parabola is clearly tangent to the r-axis. It only remains to see that it is tangent to the line $s = r$. Computing the intersection of these two curves, we see that there is only one intersection point and it occurs at $(1, 1)$. Since the line $s = r$ is not parallel to the s-axis, it follows that it must be tangent to the parabola.

2. Let $ABCD$ be a quadrilateral with AB not parallel to CD. The circle with diameter AB is tangent to the side CD at X. The circle with diameter CD is tangent to the side AB at Y.

   Prove that the quadrilateral $BCXY$ is cyclic.

Solution 1 (Norman Do)

Let M and N be the midpoints of AB and CD, respectively. Then MX is perpendicular to CD, since MX is the radius of a circle to which CD is tangent. Similarly, NY is perpendicular to AB. It follows that the points M, N, X, Y lie on a circle.



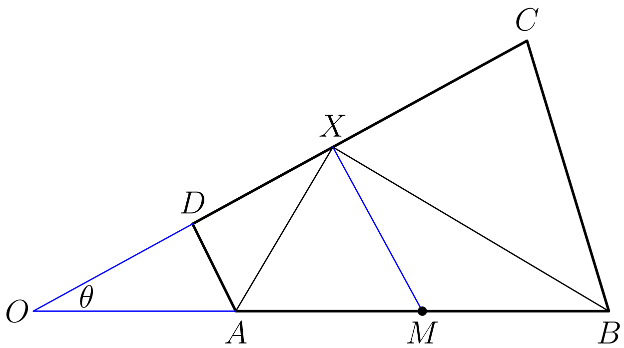
Let $\angle ABX = x$ and note that $\angle AMX = 2x$, since it is the angle subtended at the centre of the circumcircle of triangle ABX. It follows that $\angle YNX = \angle YMX = \angle AMX = 2x$, where we have used the fact that $MNXY$ is a cyclic quadrilateral.

So $\angle CNY = 180° - \angle YNX = 180° - 2x$. However, note that triangle CNY is isosceles with $CN = NY$. Therefore, $\angle NCY = \angle NYC = x$. Since $\angle XCY = \angle XBY = x$, we have deduced that the quadrilateral $BCXY$ is cyclic.

A second case arises when X and Y are on different sides of the line MN, in which case we have the equality $\angle YNX = 180° - \angle YMX$ rather than $\angle YNX = \angle YMX$. This can be handled in an analogous manner or with the use of directed angles.

Solution 2 (Norman Do)

Suppose that the lines AB and CD meet at O, and let $\angle AOD = \theta$. If M is the midpoint of AB, then the angle sum in right-angled triangle MXO yields $\angle XMO = 90° - \theta$. Therefore, the angle subtended by AX in the circle with diameter AB is $\angle ABX = \frac{1}{2}\angle AMX = 45° - \frac{\theta}{2}$.



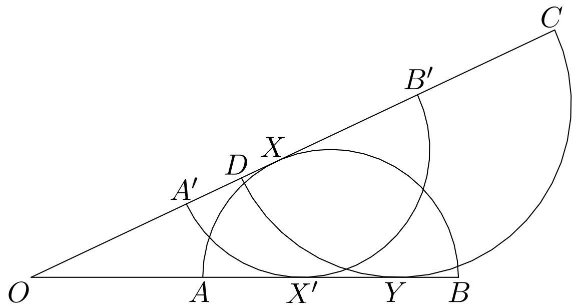
An analogous calculation using the circle with diameter CD yields $\angle DCY = 45^\circ - \frac{\theta}{2}$ as well. Since $\angle XBY = \angle XCY = 45^\circ - \frac{\theta}{2}$, we have deduced that the quadrilateral $BCXY$ is cyclic.

Solution 3 (Angelo Di Pasquale)

Suppose that the lines AB and CD meet at O. The reflection of the semicircle AXB in the bisector of $\angle AOD$, results in a semicircle $A'X'B'$, as shown in the diagram below. This semicircle is a dilation of the semicircle DYC with centre of dilation O. So using the fact that $OB' = OB$ and $OX' = OX$, we find that

$$\frac{OB'}{OC} = \frac{OX'}{OY} \quad \Rightarrow \quad \frac{OB}{OC} = \frac{OX}{OY} \quad \Rightarrow \quad OB \cdot OY = OC \cdot OX.$$

By the power of a point theorem, this implies that the quadrilateral $BCXY$ is cyclic.



Solution 4 (Angelo Di Pasquale)

Suppose that the lines AB and CD meet at a point O. There is an orientation-reversing similarity transformation that sends semicircle AXB to semicircle DYC. Since AB is not parallel to CD, it is the composition of the reflection in the bisector of $\angle AOD$ followed by the dilation by factor $\frac{DC}{AB}$ with centre O.

This implies that triangle ABX is similar to triangle DCY. Hence, $\angle OBX = \angle OCY$ and it follows that the quadrilateral $BCXY$ is cyclic.

Solution 5 (Ivan Guo)

Suppose that BX and CY intersect at Z. By considering the angle sum in triangles BYZ and CXZ, we have $\angle BXC + \angle YCX = \angle CYB + \angle XBY$. Now we invoke the alternate segment theorem to rewrite this as $\angle XAB + \angle YCX = \angle YDC + \angle XBY$. Using the fact that AB and CD are diameters, we have $90^\circ - \angle XBA + \angle YCX = 90^\circ - \angle YCD + \angle XBY$, which simplifies to $\angle YCX = \angle XBY$. Therefore, the quadrilateral $BCXY$ is cyclic.

Solution 6 (Daniel Mathews)

Let $\angle CXB = a$, $\angle XBY = b$, $\angle XCY = c$ and $\angle CYB = d$. We will show that $a = d$ and $b = c$, from which it follows that the quadrilateral $BCXY$ is cyclic.

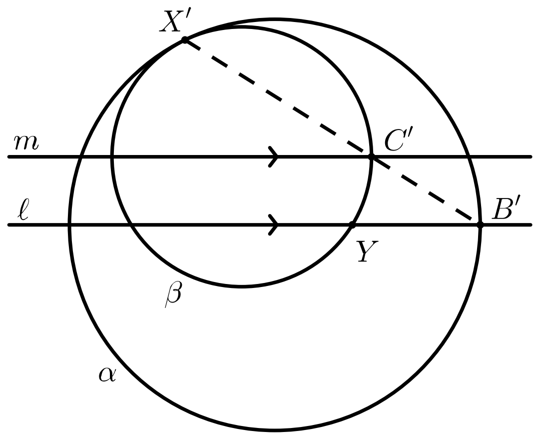
Since $180° - a - c = 180° - b - d$ is the angle between BX and CY, we have $a + c = b + d$. We have $a = \angle CXB = \angle XAB$ by the alternate segment theorem, and we also have $b = \angle XBY = 90° - \angle XAB$ from the right-angled triangle ABX. Therefore, $a + b = 90°$.

By the same argument, we have $d = \angle CYB = \angle CDY$ by the alternate segment theorem, and we also have $c = \angle XCY = 90° - \angle CDY$ from the right-angled triangle CDY. Therefore, $c + d = 90°$.

We now have $a + b = 90°$, $c + d = 90°$ and $a + c = b + d$, from which it follows that $a + c = b + d = 90°$. Hence, $a = 90° - b = d$ and $b = 90° - a = c$, giving the desired result.

Solution 7 (Alan Offer)

Consider the effect of an inversion about a circle centred at Y: the line ℓ through A and B is fixed; the circle with diameter AB maps to a circle α with centre on ℓ; the circle with diameter CD maps to a line m parallel to ℓ; the line through C and D maps to a circle β through Y with centre on m and internally tangent to α at X', the image of X; and the images B' and C' of B and C, respectively, lie in the same direction from X' on their respective circles.



Now circle α is related to β by a dilation about X', which maps m through the centre of β to the parallel line ℓ through the centre of α, so C' is mapped to B'. Hence, B', C' and X' are collinear, which under the inversion reveals that $BCXY$ is cyclic.

3. Let $a_1 < a_2 < \cdots < a_{2017}$ and $b_1 < b_2 < \cdots < b_{2017}$ be positive integers such that

$$\big(2^{a_1}+1\big)\big(2^{a_2}+1\big)\cdots\big(2^{a_{2017}}+1\big) = \big(2^{b_1}+1\big)\big(2^{b_2}+1\big)\cdots\big(2^{b_{2017}}+1\big).$$

Prove that $a_i = b_i$ for $i = 1, 2, \ldots, 2017$.

Solution 1 (Norman Do)

Suppose that there is some $i \in \{1, 2, \ldots, 2017\}$ for which $a_i \neq b_i$. Then if we cancel out equal factors on both sides of the equation, we obtain an equation of the form

$$(2^{A_1}+1)(2^{A_2}+1)\cdots(2^{A_n}+1) = (2^{B_1}+1)(2^{B_2}+1)\cdots(2^{B_n}+1),$$

where we may assume that $A_1 < A_2 < \cdots < A_n$, $B_1 < B_2 < \cdots < B_n$ and $A_1 < B_1$, without loss of generality.

Expanding both sides of the equation yields an equation of the form

$$1 + 2^{A_1} + [\,\text{higher powers of } 2\,] = 1 + 2^{B_1} + [\,\text{higher powers of } 2\,],$$

from which we obtain

$$2^{A_1} + [\,\text{higher powers of } 2\,] = 2^{B_1} + [\,\text{higher powers of } 2\,].$$

However, note that 2^{B_1} divides the right hand side but not the left hand side, which yields the desired contradiction. It follows that $a_i = b_i$ for $i = 1, 2, \ldots, 2017$.

Solution 2 (Alice Devillers, Dan Mathews and Kevin McAvaney)

We will prove the following statement for all positive integers n by induction. If $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$ are positive integers such that

$$\big(2^{a_1}+1\big)\big(2^{a_2}+1\big)\cdots\big(2^{a_n}+1\big) = \big(2^{b_1}+1\big)\big(2^{b_2}+1\big)\cdots\big(2^{b_n}+1\big),$$

then $a_i = b_i$ for $i = 1, 2, \ldots, n$.

The statement is clearly true for $n = 1$, since $2^{a_1}+1 = 2^{b_1}+1$ implies that $a_1 = b_1$.

Now assume that the statement is true for $n = k-1$ where $k \geq 2$ is an integer and consider the case $n = k$. Suppose that $a_1 < a_2 < \ldots < a_k$ and $b_1 < b_2 < \ldots < b_k$ are positive integers such that

$$\big(2^{a_1}+1\big)\big(2^{a_2}+1\big)\cdots\big(2^{a_k}+1\big) = \big(2^{b_1}+1\big)\big(2^{b_2}+1\big)\cdots\big(2^{b_k}+1\big).$$

If $a_1 \neq b_1$, we may assume without loss of generality that $a_1 < b_1$. We know that $a_i \geq a_1+1$ for all $2 \leq i \leq k$ and $b_j \geq a_1 + 1$ for all $1 \leq j \leq k$. Now consider the equation above modulo 2^{a_1+1}. The left side is $2^{a_1}+1$, while the right side is 1, which yields the desired contradiction.

So we have deduced that $a_1 = b_1$ and hence,

$$\big(2^{a_2}+1\big)\big(2^{a_3}+1\big)\cdots\big(2^{a_k}+1\big) = \big(2^{b_2}+1\big)\big(2^{b_3}+1\big)\cdots\big(2^{b_k}+1\big)$$

with $a_2 < a_3 < \cdots < a_k$ and $b_2 < b_3 < \cdots < b_k$. By the induction hypothesis, we know that $a_i = b_i$ for all $2 \leq i \leq k$. This completes the proof of the statement by induction and we recover the original problem in the case $n = 2017$.

4. Find all positive integers $n \geq 5$ for which we can place a real number at each vertex of a regular n-sided polygon, such that the following two conditions are satisfied.

   - None of the n numbers is equal to 1.
   - For each vertex of the polygon, the sum of the numbers at the nearest four vertices is equal to 4.

Solution 1 (Angelo Di Pasquale)

The answer is any even $n \geq 6$.

If n is even, the conditions are clearly satisfied if we alternate $0.5, 1.5, 0.5, 1.5, \ldots$ around the polygon.

Now suppose that $n = 2m+1$ is odd and let the numbers be $x_1, x_2, \ldots, x_n$ in order around the polygon. Here and throughout this proof, we consider the subscripts modulo n. Then for each i, we have

$$\begin{aligned} & x_i + x_{i+1} + x_{i+3} + x_{i+4} = x_{i+1} + x_{i+2} + x_{i+4} + x_{i+5} \\ \Rightarrow \quad & x_i + x_{i+3} = x_{i+2} + x_{i+5}. \qquad (*) \end{aligned}$$

For each i, let $A_i = x_i + x_{i+3}$. Then equation (*) may be written as $A_i = A_{i+2}$. Thus, the sequence $A_1, A_2, A_3, \ldots$ has period 2. However, it also has period n and hence, it has period $\gcd(2, n) = 1$. In particular, we have $A_i = A_{i+3}$, which implies that $x_i = x_{i+6}$.

So we have deduced that the sequence x_1, x_2, x_3, . . . has period 6. However, it also has period n and hence, it has period $\gcd(6, n) = 1$ or 3. In either case, equation (*) simplifies to $x_i = x_{i+2}$. So the sequence x_1, x_2, x_3, . . . has period 2. However, it also has period n and hence, it has period $\gcd(2, n) = 1$. It follows that all of the x_i are equal to 1, which yields the desired contradiction.

Solution 2 (Dan Mathews and Ian Wanless)

The answer is any even $n \geq 6$.

If n is even, the conditions are clearly satisfied if we alternate $0.5, 1.5, 0.5, 1.5, \ldots$ around the polygon.

Now suppose that $n = 2m+1$ is odd and let the numbers be $x_1, x_2, \ldots, x_n$ in order around the polygon. Here and throughout this proof, we consider the subscripts modulo n.

Define $y_i = x_i + x_{i+1}$ and observe that by the given conditions, we have $y_i = 4 - y_{i+3}$ for each i. By repeated application of this rule, we can deduce that $y_i = 4 - y_{i+3n}$, since n is odd. However, we have $y_{i+3n} = y_i$, so it follows that $y_i = 2$ for all i.

Hence, $x_i + x_{i+1} = 2 = x_{i+1} + x_{i+2}$, from which we deduce that $x_i = x_{i+2}$. So the sequence $x_1, x_2, x_3, \ldots$ has period 2. However, it also has period n and hence, it has period $\gcd(2, n) = 1$. It follows that all of the x_i are equal to 1, which yields the desired contradiction.

5. Let n be a positive integer. Consider $2n$ points equally spaced around a circle. Suppose that n of the points are coloured blue and the remaining n points are coloured red. We write down the distance between each pair of blue points in a list, from shortest to longest. We write down the distance between each pair of red points in another list, from shortest to longest. (Note that the same distance may occur more than once in a list.)

   Prove that the two lists of distances are the same.

Solution 1 (Kevin McAvaney)

The distance between two of the points is uniquely determined by the number of points between them on the circle. So if two of the points have $k-1$ points between them where $k \leq n$, we say that their *chord length* is k.

If n consecutive points are coloured blue, then the remaining n consecutive points are coloured red. Due to the symmetry of this configuration, the two lists of distances are the same.

If there are no n consecutive red points, then one can obtain n consecutive red points by repeatedly switching colours on adjacent pairs of points. We show that the lists of chord lengths are the same after one such switch if and only if they are the same before the switch.

Consider a pair of adjacent points X and Y, where X is red and Y is blue. Draw a diameter of the circle perpendicular to XY. For each point U on the same side of the diameter as X, there is a corresponding point V on the same side of the diameter as Y such that the chord lengths XU and YV are equal to k for some $1 \leq k \leq n-2$.

For each $1 \leq k \leq n-2$, there are four possibilities for the colours of U and V — namely, red-red, red-blue, blue-red and blue-blue.

- In the first case, a red-red chord of length k is changed to a red-red chord of length $k+1$ and a red-red chord of length $k+1$ is changed to a red-red chord of length k.
- In the second case, a red-red chord of length k is changed to a red-red chord of length $k+1$ and a blue-blue chord of length k is changed to a blue-blue chord of length $k+1$.
- In the third case, a red-red chord of length $k+1$ is changed to a red-red chord of length k and a blue-blue chord of length $k+1$ is changed to a blue-blue chord of length k.
- In the fourth case, a blue-blue chord of length k is changed to a blue-blue chord of length $k+1$ and a blue-blue chord of length $k+1$ is changed to a blue-blue chord of length k.

Thus, the lists of chord lengths are the same after a switch if and only if they are the same before the switch. It follows that the lists of distances are the same for any colouring.

Solution 2 (Alice Devillers, Kevin McAvaney and Ian Wanless)

We use the notion of chord length defined in Solution 1.

Let b_k the number of pairs of blue points whose chord length is k. Let r_k be the number of pairs of red points whose chord length is k. Let m_k be the number of pairs of points, one blue and one red, whose chord length is k.

For $1 \leq k < n$, the number of blue points is equal to $\frac{2b_k+m_k}{2} = n$. Note that we divide by 2 here, as each point is a member of two pairs whose chord length is k. Similarly, we obtain that the number of red points is equal to $\frac{2r_k+m_k}{2} = n$. It immediately follows that $b_k = r_k$.

Furthermore, the number of blue points is equal to $2b_n + m_n = n$. Note that we do not need to divide by 2 here, as each point is a member of only one pair whose chord length is n. Similarly, we obtain that the number of red points is equal to $2r_n + m_n = n$. It immediately follows that $b_n = r_n$.

Since we have shown that $b_k = r_k$ for $1 \leq k \leq n$, it follows that the lists of distances are the same.

AMOC SENIOR CONTEST STATISTICS

Distribution of Awards/School Year

School Year	Number of Students	Gold	Silver	Bronze	HM	Participation
10	30	2	4	6	7	11
11	45	9	8	12	7	9
Other	24	2	2	6	6	8
Total	99	13	14	24	20	28

Score Distribution/Problem

Number of Students/Score

Problem Number	0	1	2	3	4	5	6	7	Mean
1	14	1	7	5	3	6	7	54	5.1
2	22	15	0	4	1	0	1	51	4.2
3	49	2	0	0	1	0	3	34	2.9
4	9	23	10	2	8	2	2	35	3.8
5	43	10	1	1	0	0	7	16	2.2

Note: These counts do not include students who did not attempt the problem.

Mean Score/Problem/School Year

School Year	Number of Students	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Overall Mean
10	30	4.5	3.6	2.2	3.2	2.1	14.7
11	45	6.0	4.6	3.7	4.5	2.5	19.9
Other	24	4.0	4.2	2.4	3.4	1.5	13.6
Total	99	5.1	4.2	2.9	3.8	2.2	16.8