AMOC SENIOR CONTEST SOLUTIONS

1. Determine the maximum possible value of $a+b$, where a and b are two different non-negative real numbers that satisfy

$$a+\sqrt{b}=b+\sqrt{a}.$$

Solution (Angelo Di Pasquale)

The equation may be rewritten as

$$\sqrt{a}-\sqrt{b}=a-b \quad \Leftrightarrow \quad \sqrt{a}-\sqrt{b}=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) \quad \Leftrightarrow \quad \sqrt{a}+\sqrt{b}=1,$$

since $a \neq b$.

Squaring the last equation and rearranging yields

$$a+b=1-2\sqrt{ab} \quad \Rightarrow \quad a+b \leq 1.$$

Since $a=0$ and $b=1$ satisfy the original equation and satisfy $a+b=1$, it follows that the maximum possible value of $a+b$ is 1.

2. Prove that, among any ten consecutive positive integers, there are five numbers such that no two of them have a common factor larger than 1.

Solution 1 (Norman Do)

Among any ten consecutive positive integers, five of them are odd and the remaining five are even.

- The five odd integers contain either one or two multiples of 3. We remove one multiple of 3 and select the remaining four integers.
- The five even integers contain at most two multiples of 3, at most one multiple of 5, and at most one multiple of 7. Therefore, there is at least one of them that is not a multiple of 3, 5 or 7. We select this integer.

We now prove that, among the five integers selected, no two of them have a common factor larger than 1. Since the largest possible difference between two of the numbers is 9, the largest possible common factor that two of the numbers can have is 9. So it suffices to show that we have not selected a pair of numbers with a common factor of 2, 3, 5 or 7.

- By construction, we have chosen exactly one number that is divisible by 2.
- By construction, we have chosen at most one number that is divisible by 3.
- At most one of the odd integers selected is divisible by 5 and, by construction, the even integer selected is not divisible by 5.
- At most one of the odd integers selected is divisible by 7 and, by construction, the even integer selected is not divisible by 7.

Therefore, we have selected five numbers such that no two of them have a common factor larger than 1.

Solution 2 (Alice Devillers, Angelo Di Pasquale and Daniel Mathews)

Among the ten consecutive positive integers, let n be the smallest number that is not divisible by 2 or 3. Either the smallest or the second smallest odd number among the ten numbers satisfies this condition. So n is one of the four smallest integers and the five numbers $n, n+2, n+3, n+4, n+6$ are all among the ten consecutive integers.

We proceed to calculate all possible greatest common divisors between two of these five numbers. We begin with the observation that for any integer n, we have

$$\gcd(n+2, n+3) = \gcd(n+3, n+4) = 1.$$

Since n is not divisible by 2, we have

$$\gcd(n, n+2) = \gcd(n, n+4) = \gcd(n+2, n+4) = \gcd(n+2, n+6) = \gcd(n+4, n+6) = 1.$$

Since n is not divisible by 3, we have

$$\gcd(n, n+3) = \gcd(n+3, n+6) = 1.$$

Since n is not divisible by 2 or 3, we have

$$\gcd(n, n+6) = 1.$$

This exhausts all possibilities, so no two of the five numbers $n, n+2, n+3, n+4, n+6$ have a common factor larger than 1.

Solution 3 (Kevin McAvaney)

Any common factor of two integers must divide their difference. So any common factor of two of the ten integers is at most 9. Therefore, for such a common factor to be greater than 1, it must have 2, 3, 5 or 7 as a prime factor.

The sequence of remainders of the ten consecutive integers after division by 2 must be one of

0101010101 or 1010101010.

The sequence of remainders of the ten consecutive integers after division by 3 must be one of

0120120120 or 1201201201 or 2012012012.

For the resulting six cases, we choose the five numbers from among the ten original integers corresponding to the underlined remainders shown below.

$$\begin{array}{ccc}
0101010101 & 0101010101 & 0101010101 \\
0\underline{1}2\underline{0}\underline{1}\underline{2}0\underline{1}20 & 1\underline{2}0\underline{1}2\underline{0}1\underline{2}0\underline{1} & 201\underline{2}0\underline{1}\underline{2}\underline{0}1\underline{2} \\
 & & \\
1010101010 & 1010101010 & 1010101010 \\
01\underline{2}0\underline{1}\underline{2}\underline{0}1\underline{2}0 & \underline{1}2\underline{0}\underline{1}\underline{2}0\underline{1}201 & \underline{2}0\underline{1}2\underline{0}1\underline{2}0\underline{1}2
\end{array}$$

One can check that in each of the six cases, none of the primes 2, 3, 5 or 7 is a factor of more than one of the chosen integers. So in each case, no two of the five chosen numbers have a common factor lager than 1.

3. Fourteen people meet one day to play three matches of netball. For each match, they divide themselves into two teams of seven players. In each match, one team wins while the other team loses. After all three matches, no person has been on a losing team three times.

   Prove that there are at least three players who were on the same team as each other for all three matches.

Solution 1 (Norman Do)

For each person, we record the results of their matches with a string of Ws and Ls, where W denotes a win and L denotes a loss. The record of each player's results is one of the following seven possibilities.

$$WWW \qquad WWL \qquad WLW \qquad WLL \qquad LWW \qquad LWL \qquad LLW$$

It is impossible for each of these records to be obtained by exactly two players. That would imply that the sum of the number of wins for each player is 24, while the sum of the number of losses for each player is 18. However, there should be an equal number of wins and losses overall.

Hence, there must exist three players who have the same record. It follows that these three players were on the same team as each other for all three matches.

Solution 2 (Alice Devillers and Ivan Guo)

By renaming players, we can assume that the losing players in match 1 are $1, 2, 3, 4, 5, 6, 7$. The conditions of the problem imply that players $1, 2, 3, 4, 5, 6, 7$ must be on the winning team in at least one of match 2 or match 3.

By renaming players $1, 2, 3, 4, 5, 6, 7$ again and swapping the names of matches 2 and 3, we can assume by the pigeonhole principle that players $1, 2, 3, 4$ won match 2.

If player 5 also won match 2, then the pigeonhole principle guarantees that at least three of the players $1, 2, 3, 4, 5$ were on the same team for match 3 as well. Then these three players were on the same team as each other for all three matches, which is what we wanted to prove. The same argument may also be applied if player 6 or player 7 won match 2.

The only remaining case to analyse is if players $5, 6, 7$ all lost match 2. But in that case, they must all have won match 3 and were on the same team as each other for all three matches.

Solution 3 (Chaitanya Rao and Ian Wanless)

Consider the seven players who lost the first match. Then the record of each of these player's results for matches 2 and 3 must be WW, WL or LW. By the pigeonhole principle, at least three of these seven players had the same record for matches 2 and 3. Therefore, these three players were on the same team as each other for all three matches.

Solution 4 (Thanom Shaw)

The 14 people play 3 matches each, which implies that there is a total of 21 wins and 21 losses between them. Suppose that the 21 losses are obtained by x people recording 1 loss overall, and y people recording 2 losses overall. Since no one loses all 3 matches, this yields $x + 2y = 21$, where $x + y \leq 14$.

The only integer solutions for (x, y) are $(1, 10)$, $(3, 9)$, $(5, 8)$ and $(7, 7)$. Note that in each case, we have $y \geq 7$. That is, 7 or more people recorded 2 losses overall. There are only three ways to record 2 losses overall — namely, WLL, LWL and LLW. Since at least 7 people had one of these three win–loss records, the pigeonhole principle asserts that at least 3 of them had the same win–loss record. It follows that these three players were on the same team for all three matches.

4. Let K_1 and K_2 be circles that intersect at two points A and B. The tangents to K_1 at A and B intersect at a point P inside K_2, and the line BP intersects K_2 again at C. The tangents to K_2 at A and C intersect at a point Q, and the line QA intersects K_1 again at D.

Prove that QP is perpendicular to PD if and only if the centre of K_2 lies on K_1.

Solution (Andrew Elvey Price)

We first prove that if the centre O of K_2 lies on K_1, then QP is perpendicular to PD. Since DA is tangent to K_2, we have $\angle DAO = 90°$, so OD is a diameter of K_1. So by symmetry, D, O and P all lie on the line joining the centres of K_1 and K_2. Observe that A is the reflection of B in this line. Hence, $\angle PAO = \angle PBO$, but $\angle PBO = \angle CBO = \angle BCO$, since triangle BOC is isosceles. It follows that $AOPC$ is a cyclic quadrilateral. Moreover, $\angle OAQ = \angle OCQ = 90°$, so $OAQC$ is also a cyclic quadrilateral. Therefore, $OAQCP$ is a cyclic pentagon and $\angle QPO = 90°$, as required.

We now prove that if QP is perpendicular to PD, then the centre O of K_2 lies on K_1. By the alternate segment theorem, we have $\angle QCA = \angle CBA$, and it follows that

$$\angle QAC = \angle QCA = \angle PAB = \angle PBA.$$

Let this angle be α. Then $\angle APB = 180° - 2\alpha = \angle AQC$, so quadrilateral $AQCP$ is cyclic. Hence, $\angle APQ = \alpha = \angle CPQ$ and we have $\angle APD = \angle BPD = 90° - \alpha$. Therefore, D, O, P and the centre X of K_1 are collinear. Since $\angle DAO = 90°$, we can deduce that $\angle XOA = 90° - \angle XDA = 90° - \angle XAD = \angle XAO$, so $XO = XA$. It follows that O lies on K_1, as required.

5. Determine all functions f defined for positive real numbers and taking positive real numbers as values such that

$$xf(xf(2y)) = y + xyf(x)$$

for all positive real numbers x and y.

Solution 1 (Angelo Di Pasquale)

Substitute $x = 1$ into the functional equation to obtain

$$f(f(2y)) = y(1 + f(1)).$$

Since $1 + f(1) > 0$, it follows that the right side of this equation covers all positive real numbers as y varies over the positive real numbers. Therefore, f is surjective and there exists a positive real number a such that $f(2a) = 1$.

Now substitute $y = a$ into the functional equation to obtain

$$xf(x) = a\,(1 + xf(x)) \qquad \Rightarrow \qquad f(x) = \frac{c}{x},$$

where $c = \frac{a}{1-a}$ is a positive real constant. (Note that the equation $xf(x) = a\,(1 + xf(x))$ implies that $a \neq 1$.)

Substituting $f(x) = \frac{c}{x}$ into the functional equation yields

$$2y = y(1 + c).$$

Therefore, $c = 1$ and we deduce that $f(x) = \frac{1}{x}$. It is easily verified that this is indeed a solution to the functional equation.

Solution 2 (Angelo Di Pasquale)

Replace y with $\frac{y}{2}$ in the functional equation to obtain

$$xf(xf(y)) = \frac{1}{2}y\,(1 + xf(x))\,. \qquad (*)$$

Substitute $x = 1$ into this equation to deduce that $f(f(y)) = ay$, where a is a constant.

Now replace y with $f(y)$ in equation $(*)$ to obtain

$$xf(xf(f(y))) = \frac{1}{2}f(y)(1 + xf(x)) \quad \Rightarrow \quad f(axy) = \frac{f(y)(1 + xf(x))}{2x} = \frac{f(y)}{2x} + \frac{f(x)f(y)}{2}.$$

Interchanging x with y in this equation yields

$$f(ayx) = \frac{f(x)}{2y} + \frac{f(x)f(y)}{2}.$$

Equating the previous two expressions for $f(axy)$ implies that

$$\frac{f(y)}{2x} = \frac{f(x)}{2y}.$$

Now substitute $y = 1$ into this equation to deduce that

$$f(x) = \frac{c}{x},$$

where $c = f(1)$ is a constant. We may now finish the proof as in the previous solution.

AMOC SENIOR CONTEST RESULTS

Name	School	Year
	Perfect Score and Gold	
James Bang	Baulkham Hills High School NSW	11
Andres Buritica	Scotch College VIC	9
Yasiru Jayasooriya	James Ruse Agricultural High School NSW	10
Sharvil Kesarwani	Merewether High School NSW	11
Preet Patel	Vermont Secondary College VIC	11
William Steinberg	Scotch College WA	10
Hadyn Tang	Trinity Grammar School VIC	9
Fengshuo (Fredy) Ye (Yip)	Knox Grammar School NSW	8
Ziqi Yuan	Narrabundah ACT	11
	Gold	
Haowen Gao	Knox Grammar School NSW	11
Ken Gene Quah	Melbourne High School VIC	10
	Silver	
Zefeng (Jeff) Li	Caulfield Grammar School, Caulfield Campus VIC	11
Junhua Chen	Caulfield Grammar School, Wheelers Hill VIC	10
Grace He	Methodist Ladies' College VIC	10
Mikhail Savkin	Gosford High School NSW	10
Harry Zhang	Christian Brothers College VIC	10
Frank Zhao	Geelong Grammar School VIC	11
David Lee	James Ruse Agricultural High School NSW	11
Zijin (Aaron) Xu	Caulfield Grammar School, Wheelers Hill VIC	10
Liam Coy	Sydney Grammar School NSW	10
Anthony Pisani	St Paul's Anglican Grammar VIC	11
Jason Wang	Queensland Academy for Science, Mathematics and Technology QLD	10
Marcus Rees	Hobart College TAS	11
Daniel Wiese	Scotch College WA	10
	Bronze	
Christopher Leak	Perth Modern School WA	10
Huxley Berry	Perth Modern School WA	10
Vicky Feng	MLC School NSW	11
Evgeniya Artemova	Presbyterian Ladies' College VIC	11
Yang Zhang	St Joseph's College, Gregory Terrace QLD	10
Reef Kitaeff	Perth Modern School WA	11
Oliver Papillo	Camberwell Grammar School VIC	11
Patrick Gleeson	St Joseph's College, Gregory Terrace QLD	10
Samuel Lam	James Ruse Agricultural High School NSW	10
Leosha Trushin	Perth Modern School WA	10
Peter Vowles	Wesley College WA	11
Kevin Wu	Scotch College VIC	11
Christopher Do	Penleigh and Essendon Grammar School VIC	11
Angus Ritossa	St Peter's College SA	11

Name	School	Year
Emilie Wu	James Ruse Agricultural High School NSW	11
Eva Ge	James Ruse Agricultural High School NSW	9
Wenguan Lu	Barker College NSW	11
Claire Huang	Radford College ACT	10
Steve Wu	Prince Alfred College SA	11
Chenxiao Zhou	Caulfield Grammar School, Wheelers Hill VIC	9
Lachlan Rowe	Canberra College ACT	11
Micah Sinclair	Perth Modern School WA	9
Le Yao Zha	St Edmund's College ACT	10
Xiaoyu Chen	All Saints' College WA	8
Honourable Mention		
Christina Lee	James Ruse Agricultural High School NSW	10
Adrian Lo	Newington College NSW	10
David Lumsden	Scotch College VIC	8
Oliver Cheng	Hale School WA	9
Shevanka Dias	All Saints' College WA	11
Ethan Ryoo	Knox Grammar School NSW	9
Zian Shang	Scotch College VIC	7
Lucinda Xiao	Methodist Ladies' College VIC	11
Elizabeth Yevdokimov	St Ursula's College QLD	9
Matthew Cho	St Joseph's College, Gregory Terrace QLD	10
Remi Hart	All Saints' College WA	10
Linda Lu	The Mac.Robertson Girls' High School VIC	11
Andrey Lugovsky	Perth Modern School WA	11
Oliver New	Scotch College VIC	8
Angela Wang	Lauriston Girl's School VIC	9
Leo Xu	All Saints Anglican School QLD	10
William Cheah	Penleigh and Essendon Grammar School VIC	4
Anagha Kanive-Hariharan	James Ruse Agricultural High School NSW	10
Dhruv Hariharan	Knox Grammar School NSW	9
Jocelin Hon	James Ruse Agricultural High School NSW	11
Yikai Wu	Prince Alfred College SA	11
Ryan Gray	Brisbane State High School QLD	11
Mikaela Gray	Brisbane State High School QLD	9
Kento Seki	All Saints Anglican School QLD	10
Vivian Wang	James Ruse Agricultural High School NSW	10
Gen Conway	Methodist Ladies' College VIC	8
Tom Hauck	All Saints Anglican School QLD	10
Hanyuan Li	North Sydney Boys High School NSW	10
Bertrand Nheu	Perth Modern School WA	11
Benjamin Davison-Petch	Christ Church Grammar School WA	11
Zhenghao Hua	Penleigh and Essendon Grammar School VIC	8
Sam Meredith	Brisbane State High School QLD	8
Tony Teng	Concordia College SA	10
Yale Cheng	Hale School WA	11

AMOC SENIOR CONTEST STATISTICS

Score Distribution/Problem

Problem Number	Number of Students/Score								Mean
	0	1	2	3	4	5	6	7	
1	12	9	6	2	4	4	8	51	**4.9**
2	24	9	0	2	8	1	5	47	**4.3**
3	24	2	0	0	0	1	1	68	**5.1**
4	62	8	4	1	1	2	1	17	**1.6**
5	32	29	8	4	0	2	2	19	**2.2**