AUSTRALIAN MATHS TRUST

2019 AMOC Senior Contest

Tuesday, 20 August 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. For $n \geq 3$, the sequence of points $A_1, A_2, \ldots, A_n$ in the Cartesian plane has increasing x-coordinates. The line A_1A_2 has positive gradient, the line A_2A_3 has negative gradient, and the gradients continue to alternate in sign, up to the line $A_{n-1}A_n$. So the zigzag path $A_1A_2 \cdots A_n$ forms a sequence of alternating peaks and valleys at $A_2, A_3, \ldots, A_{n-1}$.

   The angle less than 180° defined by the two line segments that meet at a peak is called a *peak angle*. Similarly, the angle less than 180° defined by the two line segments that meet at a valley is called a *valley angle*. Let P be the sum of all the peak angles and let V be the sum of all the valley angles.

   Prove that if $P \leq V$, then n must be even.

2. Determine all integers that can be expressed as

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{10}}$$

   where $a_1, a_2, \ldots, a_{10}$ are non-zero integers such that no two of them have a common factor greater than 1.

3. Each unit square in a 2017×2019 grid is coloured black or white such that, in each row and in each column, the number of black squares minus the number of white squares is either 1 or -1.

   What is the maximum possible difference between the number of black squares and the number of white squares in the entire grid?

4. Let ABC be a triangle. A line parallel to BC meets the side AB at P and the side AC at Q. The line through C that is parallel to AB meets the line PQ at R. Let D be the reflection of C in the line BR.

   Prove that D lies on the circumcircle of triangle APQ if and only if $AB = BC$.

5. Determine all functions f defined for real numbers and taking real numbers as values such that

$$(x-y)f(x+y) = xf(x) - yf(y)$$

   for all real numbers x and y.