AUSTRALIAN MATHS TRUST

2019 AMOC Senior Contest

Tuesday, 20 August 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. For $n \geq 3$, the sequence of points $A_1, A_2, \ldots, A_n$ in the Cartesian plane has increasing x-coordinates. The line A_1A_2 has positive gradient, the line A_2A_3 has negative gradient, and the gradients continue to alternate in sign, up to the line $A_{n-1}A_n$. So the zigzag path $A_1A_2 \cdots A_n$ forms a sequence of alternating peaks and valleys at $A_2, A_3, \ldots, A_{n-1}$.

   The angle less than 180° defined by the two line segments that meet at a peak is called a *peak angle*. Similarly, the angle less than 180° defined by the two line segments that meet at a valley is called a *valley angle*. Let P be the sum of all the peak angles and let V be the sum of all the valley angles.

   Prove that if $P \leq V$, then n must be even.

2. Determine all integers that can be expressed as

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{10}}$$

   where $a_1, a_2, \ldots, a_{10}$ are non-zero integers such that no two of them have a common factor greater than 1.

3. Each unit square in a 2017×2019 grid is coloured black or white such that, in each row and in each column, the number of black squares minus the number of white squares is either 1 or -1.

   What is the maximum possible difference between the number of black squares and the number of white squares in the entire grid?

4. Let ABC be a triangle. A line parallel to BC meets the side AB at P and the side AC at Q. The line through C that is parallel to AB meets the line PQ at R. Let D be the reflection of C in the line BR.

   Prove that D lies on the circumcircle of triangle APQ if and only if $AB = BC$.

5. Determine all functions f defined for real numbers and taking real numbers as values such that

$$(x - y)f(x + y) = xf(x) - yf(y)$$

   for all real numbers x and y.

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2019 AMOC Senior Contest Solutions

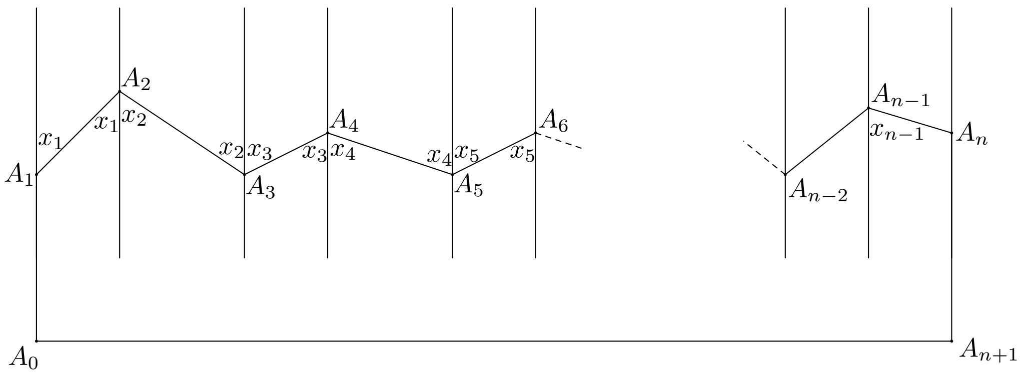
1. For $n \geq 3$, the sequence of points $A_1, A_2, \ldots, A_n$ in the Cartesian plane has increasing x-coordinates. The line A_1A_2 has positive gradient, the line A_2A_3 has negative gradient, and the gradients continue to alternate in sign, up to the line $A_{n-1}A_n$. So the zigzag path $A_1A_2 \cdots A_n$ forms a sequence of alternating peaks and valleys at $A_2, A_3, \ldots, A_{n-1}$.

   The angle less than $180°$ defined by the two line segments that meet at a peak is called a *peak angle*. Similarly, the angle less than $180°$ defined by the two line segments that meet at a valley is called a *valley angle*. Let P be the sum of all the peak angles and let V be the sum of all the valley angles.

   Prove that if $P \leq V$, then n must be even.

Solution 1 (Angelo Di Pasquale)

Assume that $P \leq V$ and that n is odd, in order to obtain a contradiction. Then $A_2, A_4, \ldots, A_{n-1}$ are peaks, while $A_3, A_5, \ldots, A_{n-2}$ are valleys.



Consider n vertical line segments, one through each A_i for $i = 1, 2, \ldots, n$. Using the fact that alternate angles are equal, we can mark the equal angles $x_1, x_2, \ldots, x_{n-1}$ as in the diagram above. Then we have the following equations.

$$P = (x_1 + x_2) + (x_3 + x_4) + \cdots + (x_{n-2} + x_{n-1})$$
$$V = (x_2 + x_3) + (x_4 + x_5) + \cdots + (x_{n-3} + x_{n-2})$$

Thus, $P = V + x_1 + x_{n-1} > V$, which yields the desired contradiction. Therefore, n must be even.

Solution 2 (Angelo Di Pasquale)

Assume that $P \leq V$ and that n is odd, in order to obtain a contradiction. Let A_0 and A_{n+1} be points that lie directly below A_1 and A_n, respectively, so that the line segment

A_0A_{n+1} is horizontal and lies below the points $A_1, A_2, \ldots, A_n$. For $i = 0, 1, 2, \ldots, n+1$, let α_i be the interior angle of the polygon $A_0A_1 \cdots A_{n+1}$ at the vertex A_i. Then we know that

$$\begin{aligned}
P &= \alpha_2 + \alpha_4 + \cdots + \alpha_{n-1},\\
V &= (360^\circ - \alpha_3) + (360^\circ - \alpha_5) + \cdots + (360^\circ - \alpha_{n-2}).
\end{aligned}$$

From the previous equation, we may deduce that

$$\alpha_3 + \alpha_5 + \cdots + \alpha_{n-2} = (n-3)180^\circ - V.$$

The sum of the interior angles of any polygon with $n+2$ sides is equal to $n \cdot 180^\circ$. Hence, we have

$$\begin{aligned}
n \cdot 180^\circ &= \alpha_0 + \alpha_1 + \cdots + \alpha_{n+1}\\
&= 90^\circ + \alpha_1 + P + (n-3)180^\circ - V + \alpha_n + 90^\circ.
\end{aligned}$$

Here, we have used the fact that $\alpha_0 = \alpha_{n+1} = 90^\circ$. It follows that

$$P = V + 360^\circ - \alpha_1 - \alpha_n.$$

But since $\alpha_1, \alpha_n < 180^\circ$, this implies that $P > V$, which yields the desired contradiction. Therefore, n must be even.

Solution 3 (Jamie Simpson and Ian Wanless)

Assume that $n = 2m+3$ is odd, so that there will be $m+1$ peaks and m valleys. Say the peak angles are $P_1, P_2, \ldots, P_{m+1}$ and the valley angles are $V_1, V_2, \ldots, V_m$. Suppose that an ant walks along the path $A_1A_2 \cdots A_n$ and that its initial direction is θ, measured clockwise from the vertical. After the first peak, its direction will be $\theta + 180^\circ - P_1$ and after the first valley, it will be

$$\theta + 180^\circ - P_1 - 180^\circ + V_1 = \theta - P_1 + V_1.$$

Continuing in the same fashion, we see that after the final peak, the ant's direction will be

$$\theta + 180^\circ - \sum_{i=1}^{m+1} P_i + \sum_{i=1}^{m} V_i = \theta + 180^\circ + V - P.$$

If $P \leq V$, this will be greater than 180°, which is impossible. Therefore, n must be even.

2. Determine all integers that can be expressed as

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{10}}$$

where $a_1, a_2, \ldots, a_{10}$ are non-zero integers such that no two of them have a common factor greater than 1.

Solution 1 (Angelo Di Pasquale)

The only integers that can be expressed in such a way are: $0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10$.

Suppose that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{10}} = n$, where $\gcd(a_i, a_j) = 1$ for all $1 \leq i < j \leq 10$. Furthermore, suppose that p is a prime that divides a_1. Multiplying the whole equation by $a_1 a_2 \cdots a_{10}$ and using $p \mid a_1$, we deduce that $p \mid a_2 a_3 \ldots a_{10}$. Hence, p divides a_i for some $i \geq 2$, which contradicts the fact that $\gcd(a_1, a_i) = 1$.

It follows that $a_1 = \pm 1$ and a similar argument shows that $a_i = \pm 1$ for $1 \leq i \leq 10$. If m of the a_i are equal to 1 and $10 - m$ of the a_i are equal to -1, then we deduce that $n = m - (10 - m) = 2m - 10$. Since m can be any integer from 0 to 10, we deduce that n can be any even integer from -10 to 10.

Solution 2 (Andrew Elvey Price)

If a prime p divides the denominator a_i of one of the fractions $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_{10}}$, then it cannot divide the denominator of any other fraction. So p cannot divide the denominator of the sum of the other fractions, when written in reduced form. However, this sum is $n - \frac{1}{a_i} = \frac{na_i - 1}{a_i}$, so the denominator must be divisible by p. This yields a contradiction.

Therefore, each a_i is not divisible by any prime and can only be equal to 1 or -1. So the only integers that can be expressed in such a way are the even numbers in the interval $[-10, 10]$.

3. Each unit square in a 2017×2019 grid is coloured black or white such that, in each row and in each column, the number of black squares minus the number of white squares is either 1 or -1.

   What is the maximum possible difference between the number of black squares and the number of white squares in the entire grid?

Solution 1 (Norman Do)

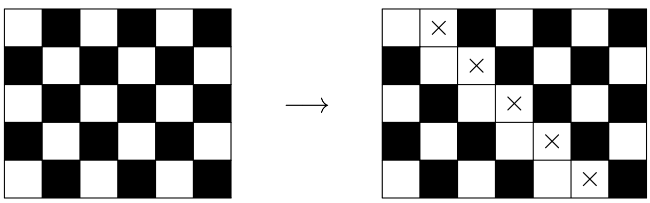
For every row and column, label it with the letter B if it contains more black squares and with the letter W if it contains more white squares. Observe that the difference between the number of black squares and white squares in the entire grid is equal to the difference between the number of B-rows and the number of W-rows. It is clear that this cannot be greater than 2017.

We will now show that it is possible for the difference between the number of black squares and white squares in the entire grid to be 2017. Start with a 2017×2017 grid of unit squares. We colour the top 1009 squares of the leftmost column black and the bottom 1008 squares white. For the next column along, we cyclically shift the colouring by one square down. We keep moving to the next column along and cyclically shifting the colouring by one square down until we have coloured the entire grid. In this colouring of the 2017×2017 grid, every row and column has one more black square than white square. We then add two more columns to the right of the grid. We colour the top 1009 squares of one column black and the bottom 1008 squares white, while we colour the top 1009 squares of the other column white and the bottom 1008 squares black. In this particular colouring of the 2017×2019 grid, the difference between the number of black squares and white squares in each row and column is 1. Furthermore, the difference between the number of black squares and white squares in the entire grid is 2017.

Solution 2 (Angelo Di Pasquale)

As in the first solution above, the answer is at most 2017. The following is an alternative construction that achieves this bound.

Colour the squares of the left 2017×2018 subgrid in the standard chessboard style. Then crack open the grid just above its main diagonal into two parts so that the lower part retains the squares on the main diagonal, but the upper part doesn't. Next move the upper part one unit to the right. We now have a 2017×2019 grid with all squares coloured except for those on the superdiagonal just formed. Colouring all the squares on the superdiagonal white yields a valid colouring. The diagram below illustrates the 5×7 case, where the marked squares are the inserted white squares.



Solution 3 (Kevin McAvaney)

As in the first solution above, the answer is at most 2017. The following is an alternative construction that achieves this bound.

In a grid with $2n-1$ rows and $2n+1$ columns, colour all squares in the top left $n\times(n+1)$ grid black, all squares in the top right $n\times n$ grid white, all squares in the bottom left $(n-1)\times(n+1)$ grid white, and all squares in the bottom right $(n-1)\times n$ grid black.

Then

- each of the leftmost $(n+1)$ columns has one more black square than white square;
- each of the rightmost n columns has one more white square than black square;
- each of the top n rows has one more black square than white square; and
- each of the bottom $(n-1)$ rows has one more white square than black square.

Starting at the top left square of the $(2n-1)\times(2n+1)$ grid and working down the main diagonal for n rows, convert each black square to a white square. Then the difference between the number of black squares and white squares in each column is still 1, but every row has one more white square than black square. So there are $2n-1$ more white squares than black squares. The desired result is recovered by taking $n=1009$.

Solution 4 (Alan Offer)

We shall address the problem for $(2n-1)\times(2n+1)$ grids, where n is a positive integer. Without loss of generality, we may suppose the black squares outnumber the white squares. That $2n-1$ is an upper bound is seen by observing that in each of the $2n-1$ rows, the number of black squares can exceed the number of white squares by at most 1. In fact, this upper bound is attainable, as we shall demonstrate, so the answer to the posed question is 2017.

We prove by induction that a $(2n-1)\times(2n+1)$ grid has such a colouring, which, more specifically, has one more white square than black squares in the first column, and one more black than white in the remaining $2n$ columns. Notice that each row necessarily has one more black than white squares.

Suppose the claim is true for some n, and let G be a $(2n-1)\times(2n+1)$ grid with such a colouring. On the right of G, append a $(2n-1)\times 2$ grid that has the pattern BW in the first n rows, and the pattern WB in the remaining $n-1$ rows. Then the number of black squares remains one more than the number of white squares in each row, and in these two new columns, the left column has one more black than white, while the right column has one more white than black.

Now below G, append a $2\times(2n+1)$ grid that has the pattern $\substack{B\\W}$ in the first $n+1$ columns, and the pattern $\substack{W\\B}$ in the remaining n columns. Then the number of black squares remains one fewer than white squares in the first column and one more in the other columns. In these two new rows, the upper row has one more black than white, while the lower rower has one more white than black.

To complete the construction, below and to the right of G append a 2×2 grid with the pattern $\begin{smallmatrix} W & B \\ B & B \end{smallmatrix}$. Then in the completed $(2n+1) \times (2n+3)$ grid, the bottom two rows and the rightmost two columns each has one more black than white squares, as required.

Finally, colouring the 1×3 grid with WBB shows that such a colouring is possible for $n = 1$, so by induction, the claim holds for all $n \geq 1$.

4. Let ABC be a triangle. A line parallel to BC meets the side AB at P and the side AC at Q. The line through C that is parallel to AB meets the line PQ at R. Let D be the reflection of C in the line BR.

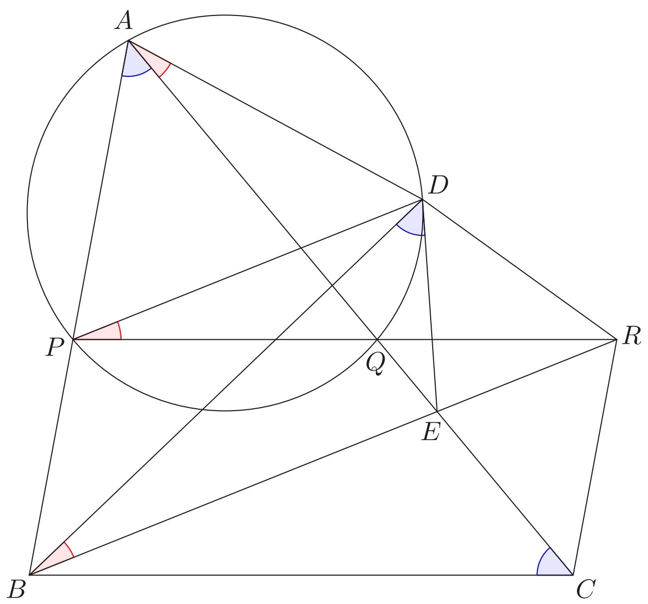
   Prove that D lies on the circumcircle of triangle APQ if and only if $AB = BC$.

Solution 1 (Angelo Di Pasquale)

Let E be the intersection of lines BR and AC. The conclusion shall follow from the following equivalences.

$$ADEB \text{ cyclic} \quad \Leftrightarrow \quad AB = BC \tag{1}$$
$$ADEB \text{ cyclic} \quad \Leftrightarrow \quad APQD \text{ cyclic} \tag{2}$$

Proof of (1). From the reflection, we have $\angle BDE = \angle ECB$. Hence, $\angle BAE = \angle BDE$ if and only if $\angle BAE = \angle ECB$. That is, $ADEB$ is cyclic if and only if $AB = BC$.



Proof of (2). From the reflection and the parallelogram $BCRP$, we have

$$\triangle BDR \equiv \triangle BCR \equiv \triangle RPB.$$

It follows that $BPDR$ is an isosceles trapezium and hence cyclic.[1] Thus, $\angle QPD = \angle RBD$. It follows that $\angle EAD = \angle EBD$ if and only if $\angle EAD = \angle QPD$. That is, $ADEB$ is cyclic if and only if $APQD$ is cyclic.

Solution 2 (Ivan Guo)

We use the diagram and notation of the first solution above. We invoke Miquel's theorem with respect to the lines AB, BR, PR, AE. In particular, the theorem states that the four

[1] From here on, the proof is just the pivot theorem applied to $\triangle EQR$ with circles EBA, QAP, and RPB .

circles APQ, ABE, RQE and RPB all pass through a common point, which is D. We analyse the four cyclic quadrilaterals one by one.

(1) $APQD$ cyclic.
This is one side of the required equivalence.

(2) $ABED$ cyclic.
This is equivalent to $AB = BC$, via the angle chase $\angle BDE = \angle BCE = \angle BAC$, as in the first solution.

(3) $RQED$ cyclic.
This is also equivalent to $AB = BC$, via the angle chase $\angle RDE = \angle RCE = \angle BAC = \angle BCA = \angle RQE$.

(4) $RPBD$ cyclic.
This is true unconditionally from the parallelogram $BCRP$ and the reflection.

Note that we want to prove (1) $\iff$ $AB = BC$. Since only two of the four circles are needed to uniquely determine D, and (4) is always true, there are two proofs as follows.

$$AB = BC \iff (2) \iff (2)+(4) \iff (1)+(4) \iff (1)$$
$$AB = BC \iff (3) \iff (3)+(4) \iff (1)+(4) \iff (1)$$

Solution 3 (Kevin McAvaney)

From the parallelogram $BCRP$ and reflection in the line BR, triangles BDP and RPD are congruent, so $\angle RPD = \angle BDP$. Let BD and PR intersect at F. Then $\angle PFB = \angle CBF$, since BC and PQ are parallel. By reflection, $\angle CBD = 2\angle RBD$. Since $\angle BFP = 2\angle FPD$, we have $\angle RPD = \angle RBD$. Again by reflection, $\angle BCE = \angle BDE$.

So

$$\begin{aligned}
& AB = AC \\
\Leftrightarrow\ & \angle BAC = \angle ACB \\
\Leftrightarrow\ & \angle BAE = \angle BDE \\
\Leftrightarrow\ & \angle EBD = \angle EAD \\
\Leftrightarrow\ & \angle QAD = \angle QPD \\
\Leftrightarrow\ & QPAD \text{ is cyclic.}
\end{aligned}$$

Solution 4 (Alan Offer)

Let BR lie along the real axis with B at the origin, so $B = 0$ and R is real. Let the scale be chosen such that $C\bar{C} = 1$. We must then show that A, P, Q and D are concyclic if and only if $A\bar{A} = 1$. We use the fact that these four points are concyclic if and only if the cross-ratio $\frac{A-P}{A-Q} / \frac{D-P}{D-Q}$ is real, which is in turn equivalent to $(A-P)(\bar{A}-\bar{Q})(D-Q)(\bar{D}-\bar{P})$ being real.

Since A, B and C are not collinear, the line BR lies strictly between BA and BC, so A is not real. For some real s, we have $P = sA$ and $Q = sA + (1-s)C$. As $BCRP$ is a parallelogram and R is real, we have $R = P + C = sA + C = s\bar{A} + \bar{C}$. Notice $Q = R - sC$. As D is the reflection of C about BR, we have $D = \bar{C}$.

Working through each factor of the cross-ratio, first we have $A - P = A - sA = (1-s)A$. Since triangles APQ and ABC are similar, we have $\bar{A} - \bar{Q} = (1-s)(\bar{A} - \bar{C})$. Next, $D - Q = \bar{C} - R + sC = -s\bar{A} + sC = -s(\bar{A} - C)$. Finally, $\bar{D} - \bar{P} = C - s\bar{A} = C + \bar{C} - R$, which is real. Thus

$$\begin{aligned}
&A,\ P,\ Q \text{ and } D \text{ are concyclic}\\
\Leftrightarrow\ &(A-P)(\bar{A}-\bar{Q})(D-Q)(\bar{D}-\bar{P}) \in \mathbb{R}\\
\Leftrightarrow\ &A(\bar{A}-\bar{C})(\bar{A}-C) \in \mathbb{R}\\
\Leftrightarrow\ &A\bar{A}^2 - (C+\bar{C})A\bar{A} + A \in \mathbb{R}\\
\Leftrightarrow\ &A\bar{A}^2 + A \in \mathbb{R}\\
\Leftrightarrow\ &A\bar{A}^2 + A - A^2\bar{A} - \bar{A} = 0\\
\Leftrightarrow\ &(A\bar{A}-1)(A-\bar{A}) = 0,
\end{aligned}$$

and since A is not real, this is equivalent to $A\bar{A} = 1$, as required.

5. Determine all functions f defined for real numbers and taking real numbers as values such that

$$(x-y)f(x+y) = xf(x) - yf(y)$$

for all real numbers x and y.

Solution 1 (Ross Atkins)

Let a be any real number with $a \neq 0, 1, 2$. The substitution $y = 1$ and $x = a$ leads to the equation

$$f(a+1) = \frac{af(a) - f(1)}{a-1}.$$

The substitution $y = 1$ and $x = a+1$ allows us to write $f(a+2)$ in terms of $f(a)$ and $f(1)$ as follows.

$$\begin{aligned} f(a+2) &= \frac{(a+1)f(a+1) - f(1)}{a} \\ &= \frac{(a+1)\left(\frac{af(a)-f(1)}{a-1}\right) - f(1)}{a} \\ &= \frac{(a+1)f(a) - 2f(1)}{a-1} \end{aligned}$$

The substitution $y = 2$ and $x = a$ allows us to write $f(a+2)$ in terms of $f(a)$ and $f(2)$.

$$f(a+2) = \frac{af(a) - 2f(2)}{a-2}.$$

We can equate these two expressions for $f(a+2)$ to obtain

$$\frac{(a+1)f(a) - 2f(1)}{a-1} = f(a+2) = \frac{af(a) - 2f(2)}{a-2}.$$

Now solve for $f(a)$ to obtain $f(a) = (a-1)f(2) + (2-a)f(1)$. Therefore, we have

$$f(x) = (x-1)f(2) + (2-x)f(1), \qquad \text{for all } x \neq 0, 1, 2.$$

Note that the equation above also holds trivially for $x = 1$ and $x = 2$, and it can be observed to hold for $x = 0$ as well, by substituting $y = -3$ and $x = 3$ into the original equations. Thus,

$$f(x) = (x-1)f(2) + (2-x)f(1) \qquad \text{for all } x \in \mathbb{R}.$$

Since $f(1)$ and $f(2)$ are simply constants, it follows that f must be a linear function. Finally, we can verify that all linear functions satisfy the functional equation, by substituting $f(x) = mx + c$.

$$\begin{aligned} (x-y)f(x+y) &= (x-y)(m(x+y)+c) = mx^2 - my^2 + cx - cy \\ xf(x) - yf(y) &= x(mx+c) - y(my+c) = mx^2 + cx - my^2 - cy \end{aligned}$$

Solution 2 (Alice Devillers)

Note that if f is a solution then $f + c$ is also a solution, so up to adding a constant, we can assume that $f(0) = 0$. We immediately see that the zero function is a solution, so from now on, we assume that $f \not\equiv 0$. Hence, there exists $z \neq 0$ such that $f(z) \neq 0$. If f is a solution, then cf is also a solution, so up to multiplying by a constant, we can assume that $f(z) = z$.

Taking $x = a$ and $y = -a$, we obtain $2af(0) = 0 = af(a) + af(-a) = a(f(a) + f(-a))$, so for $a \neq 0$, we have $f(-a) = -f(a)$. Note that this equation is also true for $a = 0$, since we have $f(0) = 0$.

Taking $x = a$ and $y = z$, we obtain $(a - z)f(a + z) = af(a) - zf(z) = af(a) - z^2$, so for $a \neq z$, we have $f(a + z) = \frac{af(a)-z^2}{a-z}$.

Taking $x = -a$ and $y = a + z$, we obtain $(-2a - z)f(z) = -af(-a) - (a + z)f(a + z) = af(a) - (a + z)f(a + z)$, so for $a \neq z$, we have

$$(-2a - z)z = af(a) - (a + z)\frac{af(a) - z^2}{a - z} = \frac{-2azf(a) + az^2 + z^3}{a - z}$$

Rearranging, we obtain $azf(a) = a^2z$ for $a \neq z$, but note that this equation is also true for $a = z$.

Now $z \neq 0$, so $af(a) = a^2$, and so $f(a) = a$ for $a \neq 0$, but note that this equation is also true for $a = 0$. So $f(x) = x$ for all x, and we easily check that this function satisfies the equation.

Since we can multiply and add constants to obtain the remaining solutions, we obtain that in general $f(x) = ax + b$ for real constants a and b.

Solution 3 (Angelo Di Pasquale)

The idea is to get two different expressions for $f(x+2)$ in terms of $f(x)$. One way is to put $y = 2$, the other way is to put $y = 1$ to get $f(x + 1)$ in terms of $f(x)$ and to get $f(x + 2)$ in terms of $f(x + 1)$. Here are the details.

Put $y = 2$ into the given functional equation to find

$$(x - 2)f(x + 2) = xf(x) - 2f(2). \tag{1}$$

Put $y = 1$ into the given functional equation to find

$$(x - 1)f(x + 1) = xf(x) - f(1). \tag{2}$$

Replace x with $x + 1$ in (2) to find

$$xf(x + 2) = (x + 1)f(x + 1) - f(1) \tag{3}$$

Multiply both sides of (3) by $x - 1$ and then use (2) to find

$$\begin{aligned} x(x - 1)f(x + 2) &= (x + 1)(x - 1)f(x + 1) - (x - 1)f(1) \\ &= (x + 1)(xf(x) - f(1)) - (x - 1)f(1). \end{aligned} \tag{4}$$

Now compare the two expressions for $f(x+2)$ in (1) and (4) in such a way that we take care not to divide by zero as follows

$$\begin{aligned} x(x-1)(xf(x)-2f(2)) &= x(x-1)(x-2)f(x+2) \\ &= (x-2)((x+1)(xf(x)-f(1))-(x-1)f(1)) \\ \Leftrightarrow \qquad xf(x) &= x(2-x)f(1)+x(x-1)f(2). \end{aligned}$$

Hence, for $x \neq 0$ we deduce $f(x) = ax+b$ for constants a and b. To plug the hole at $x = 0$, put $y = -x \neq 0$ into the given functional equation to find

$$2f(0) = f(x) + f(-x) = ax + b - ax + b = 2b.$$

Hence $f(x) = ax + b$ extends also to $x = 0$.

Finally it is routine to check that $f(x) = ax + b$ satisfies the given functional equation.

Solution 4 (Ivan Guo)

Lemma. In the Cartesian plane, the points $(x, f(x)), (y, f(y))$ and $(x + y, f(x + y))$ are collinear.

Proof. It is trivial if any of $x = 0$, $y = 0$ or $x = y$ is true. Otherwise, the lemma holds because the condition of the problem rearranges to

$$\frac{f(y)-f(x)}{y-x} = \frac{f(x+y)-f(x)}{y}.$$

Now suppose that there exist non-zero reals a, b, c such that $A = (a, f(a))$, $B = (b, f(b))$ and $C = (c, f(c))$ are not collinear. Define the points $D = (b + c, f(b + c))$, $E = (a + c, f(a + c))$ and $F = (a + b, f(a + b))$. By our lemma, D lies on BC and so on. Moreover, D lies between B and C if and only if b and c have opposite sign. Again from our lemma, the lines AD, BE, CF are concurrent through the point $(a+b+c, f(a+b+c))$. By Ceva's theorem on triangle ABC, either one or three of D, E, F must lie on the interior of their respective sides. But this is a contradiction, since we either have:

(i) a, b, c all having the same sign, which means that none of D, E, F lie on the interior;
(ii) or two of a, b, c have one sign and one has another, which means that two of D, E, F lie on the interior.

Thus A, B, C must have been collinear in the first place. Since a, b, c are arbitrary non-zero reals, the entire graph of f, with the exception of $f(0)$ must lie on the same line. Finally, by using $x = -y$ in the lemma, we see that $f(0)$ also lies on that line, so the function must be linear: $f(x) = px + q$ for real constants p and q. Routine substitution shows that all such functions satisfy the equation.

Solution 5 (Dan Mathews)

We first observe that if $f(x)$ is a solution, then for any constant c, $f(x) + c$ is also a solution.

Hence, we may assume that $f(0) = 0$. Then substitute $y = -x$ to obtain

$$0 = xf(x) + xf(-x).$$

For $x \neq 0$, we then have $f(x) = -f(-x)$, so f is an odd function.

Using the oddness of f, we observe that replacing y with $-y$ in the right hand side of the original equation yields the same expression. Hence, on the left hand side the same is true, so

$$(x-y)f(x+y) = (x+y)f(x-y).$$

For $x \neq \pm y$, we then have

$$f(x+y)/(x+y) = f(x-y)/(x-y).$$

Now for any non-zero a, b, we may find x, y such that $x + y = a$ and $x - y = b$. Thus, for all non-zero a, $f(a)/a$ is a constant. So $f(x) = cx$ for some constant c.

We conclude that any solution to the original equation is a linear function; and we observe that any linear function is indeed a solution.

2019 AMOC SENIOR CONTEST STATISTICS

Score Distribution/Problem

Problem Number	Number of Students/Score								Mean
	0	1	2	3	4	5	6	7	
1	4	11	6	3	1	4	14	82	**5.7**
2	31	8	12	3	13	8	3	47	**3.8**
3	30	2	17	1	2	1	2	70	**4.4**
4	81	11	3	6	5	0	0	19	**1.5**
5	55	51	2	0	1	1	1	14	**1.3**