AUSTRALIAN MATHS TRUST

2020 AMOC Senior Contest

Tuesday, 18 August 2020
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Given real numbers a and b, prove that there exists a real number x that satisfies at least one of the following three equations.

$$x^2 + 2ax + b = 0$$
$$ax^2 + 2bx + 1 = 0$$
$$ax^2 + 2x + b = 0$$

2. Let m and n be integers greater than 1. We would like to write each of the numbers $1, 2, 3, \ldots, mn$ in the mn unit squares of an $m \times n$ chessboard, one number per square, according to the following rules.

   (i) Each pair of consecutive numbers must be written within one row or column of the chessboard.

   (ii) No three consecutive numbers can be written within one row or column of the chessboard.

   For which values of m and n is this possible?

3. Let a_1 be a given integer greater than 1. For $k = 2, 3, 4, \ldots$, let a_k be the smallest positive integer that satisfies the following conditions:

   (i) $a_k > a_{k-1}$

   (ii) a_k is not divisible by a_r for any $r < k$.

   Prove that the number of composite numbers in the sequence $a_1, a_2, a_3, \ldots$ is finite.

4. Let ABC be an acute triangle with $AB > AC$. Let O be the circumcentre of triangle ABC and P be the foot of the altitude from A to BC. Denote the midpoints of the sides BC, CA and AB by D, E and F, respectively. The line AO intersects the lines DE and DF at Q and R, respectively.

   Prove that D is the incentre of triangle PQR.

5. Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\left(x^{f(y)}\right) = f(x)^y$$

   for all positive real numbers x and y.