AMOC SENIOR CONTEST 2020

Solutions

1. Given real numbers a and b, prove that there exists a real number x that satisfies at least one of the following three equations.

$$x^2 + 2ax + b = 0$$
$$ax^2 + 2bx + 1 = 0$$
$$ax^2 + 2x + b = 0$$

Solution 1 (Norman Do)

Suppose that there does not exist a real number x that satisfies at least one of the three equations. The discriminants of the three quadratic equations are respectively $4a^2 - 4b$, $4b^2 - 4a$, and $4 - 4ab$. Therefore, we have

$$4a^2 - 4b < 0, \qquad 4b^2 - 4a < 0, \qquad 4 - 4ab < 0,$$

which simplify to

$$a^2 < b, \qquad b^2 < a, \qquad 1 < ab.$$

The first two inequalities imply that a and b are positive real numbers. Hence we may multiply all three inequalities together to obtain $a^2b^2 < a^2b^2$, which is clearly a contradiction. It follows that there must exist a real number x that satisfies at least one of the three equations.

Remark

After reaching $a^2 < b, b^2 < a, ab > 1$ and $a, b > 0$, there are many ways to finish the proof algebraically. Here are some more examples.

- (Angelo Di Pasquale) Multiplying the first two inequalities yield $a^2b^2 < ab$, which simplifies to $ab < 1$ and contradicts the third inequality.
- (Angelo Di Pasquale) From the first two inequalities we have $a^4 < b^2 < a$, which implies $a < 1$. Similarly we have $b < 1$. Hence $ab < 1$, which is a contradiciton.
- (Ian Wanless) Without loss of generality, assume $0 < a \leq b$. If $b \geq 1$ then $b^2 \geq b \geq a$, which contradicts the second inequality. So we may assume that $0 < a \leq b < 1$. But then $ab < 1$, contradicting the third inequality.

Solution 2 (Norman Do)

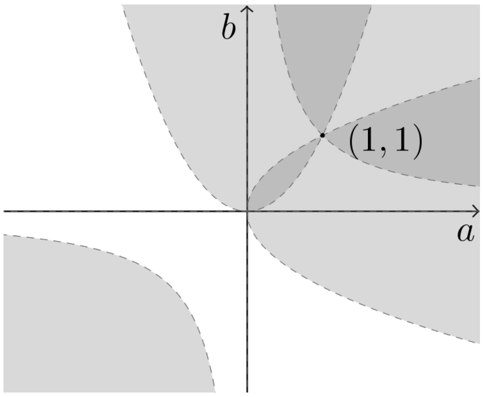
Assume no such x exists. So each discriminant is negative, which implies that the sum of the discriminants is negative. However, the sum of the discriminants is

$$2(a-1)^2 + 2(b-1)^2 + 2(a-b)^2,$$

which is necessarily non-negative. This contradiction implies that such an x exists.

Solution 3 (Alan Offer)

Following the argument of Solution 1, we reach the inequalities $a^2 < b, b^2 < a$ and $ab < 1$. Let us plot the three regions below.



Note that the three boundary curves all pass through the point $(1, 1)$, revealing that there is no point common to all three regions. Therefore there are no such values of a and b, which leads to the required contradiction.

2. Let m and n be integers greater than 1. We would like to write each of the numbers $1, 2, 3, \ldots, mn$ in the mn unit squares of an $m \times n$ chessboard, one number per square, according to the following rules.

   (i) Each pair of consecutive numbers must be written within one row or column of the chessboard.

   (ii) No three consecutive numbers can be written within one row or column of the chessboard.

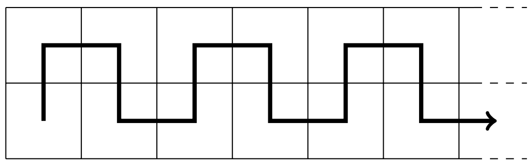
   For which values of m and n is this possible?

Solution 1 (Chris Wetherell)

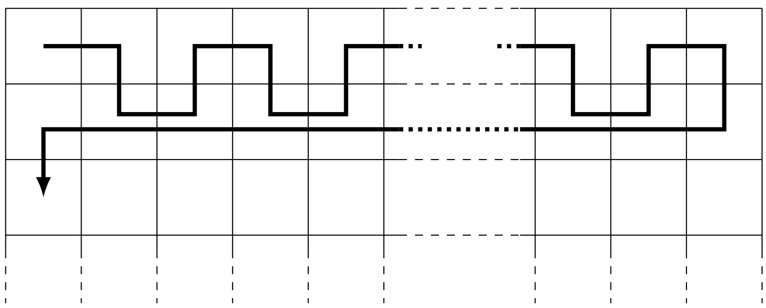
It is possible to number the squares according to the rules if and only if $m = 2$ or $n = 2$ or both m, n are even.

By interpreting the numbering as the order in which a rook is visiting the squares of the chessboard, the problem is equivalent to checking the existence of a rook tour that alternates between vertical and horizontal moves while visiting every square.

If at least one of m or n equals 2, then a rook can perform a tour of the following type,



If both m and n are even, then the rook can perform a tour by repeating the following sequence of moves in pairs of rows,



It remains to show that the above two cases are the only possibilities. Assume the contrary. Without loss of generality, suppose the number of columns m is odd, while the number of rows n is greater than 2. Then there exists some row R that does not include the starting square or the finishing square. Whenever the rook enters row R, it makes one horizontal move before exiting. So each visit to row R affects exactly two squares. Since the rook did not start or finish on row R, the number of squares in that row, m, must be even. This is a contradiction and the proof is complete.

Solution 2 (Alan Offer) Without loss of generality, let m be the number of rows. Let us construct the following complete bipartite graph with $m + n$ vertices and mn edges.

The vertices are labelled $r_1, r_2, \ldots, r_m$ and $c_1, c_2, \ldots, c_n$, which correspond to the rows and columns of the chessboard. For each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, let there be an edge joining r_i and c_j, which corresponds to the square of the chessboard in row i and column j.

By interpreting a numbering of the squares as an ordering of the edges, the existence of a numbering corresponds to the existence of an Eulerian path in this graph, that is, a path which visits every edge exactly once. It is well-known that an Eulerian path exists in a graph if and only if the number of vertices with odd degrees is at exactly 0 or 2. By construction, each r_i has degree n and each c_j has degree m, so the sequence of degrees looks like

$$\underbrace{m, \ldots, m}_{n}, \underbrace{n, \ldots, n}_{m}.$$

Having no odd degrees is equivalent to both m, n being even. Having exactly two odd degrees is equivalent to either $m = n = 1$, or one of m, n being odd and the other being 2. Since the case $m = n = 1$ is ruled out by the problem, the possible cases must be $m = 2$ or $n = 2$ or both m, n are even.

3. Let a_1 be a given integer greater than 1. For $k = 2, 3, 4, \ldots$, let a_k be the smallest positive integer that satisfies the following conditions:

   (i) $a_k > a_{k-1}$

   (ii) a_k is not divisible by a_r for any $r < k$.

   Prove that the number of composite numbers in the sequence $a_1, a_2, a_3, \ldots$ is finite.

Solution 1 (Angelo Di Pasquale)

By definition, The sequence $a_1, a_2, \ldots$ is strictly increasing. It is enough to show that if $a_i > a_1^2$ then a_i is prime.

Suppose, for the sake of contradiction, that $a_i = cd$ is composite with $c, d > 1$. Without loss of generality $c > a_1$. Then there is a unique index $k < i$ such that $a_{k-1} < c \leq a_k$. If $c = a_k$, then $a_k \mid a_i$, which is a contradiction. So c is not a term of the sequence.

Hence $a_{k-1} < c < a_k$. By definition, a_k is the smallest integer greater than a_{k-1} that is not divisible by any previous term of the sequence. Hence c must be divisible by a_r for some $r < k$. Therefore, $a_r \mid a_i$, which is a contradiction.

Solution 2 (Daniel Mathews, Kevin McAvaney, Ian Wanless)

For any prime p, define $f(p)$ to be the positive integer satisfying

$$p^{f(p)-1} < a_1 \leq p^{f(p)}.$$

Note that $p^{f(p)}$ is a term of the sequence because all of its proper divisors are not.

Let q be a prime factor of a composite term C of the sequence. If $f(q) = 1$ then, by the above, q is a term of the sequence. But then no other term of the sequence is divisible by q, contradicting our choices of C and q.

Hence $f(q) > 1$, which means that q is one of the finitely many primes less than a_1. Furthermore $q^{f(q)+1}$ does not divide C, since $q^{f(q)}$ is a term of the sequence and cannot be a proper divisor of C.

Therefore, every prime factor q of C is less than a_1 and the higher power of q that divides C is at most $q^{f(q)}$. So C must be bounded above by

$$q_1^{f(q_1)} q_2^{f(q_2)} \cdots q_m^{f(q_m)}$$

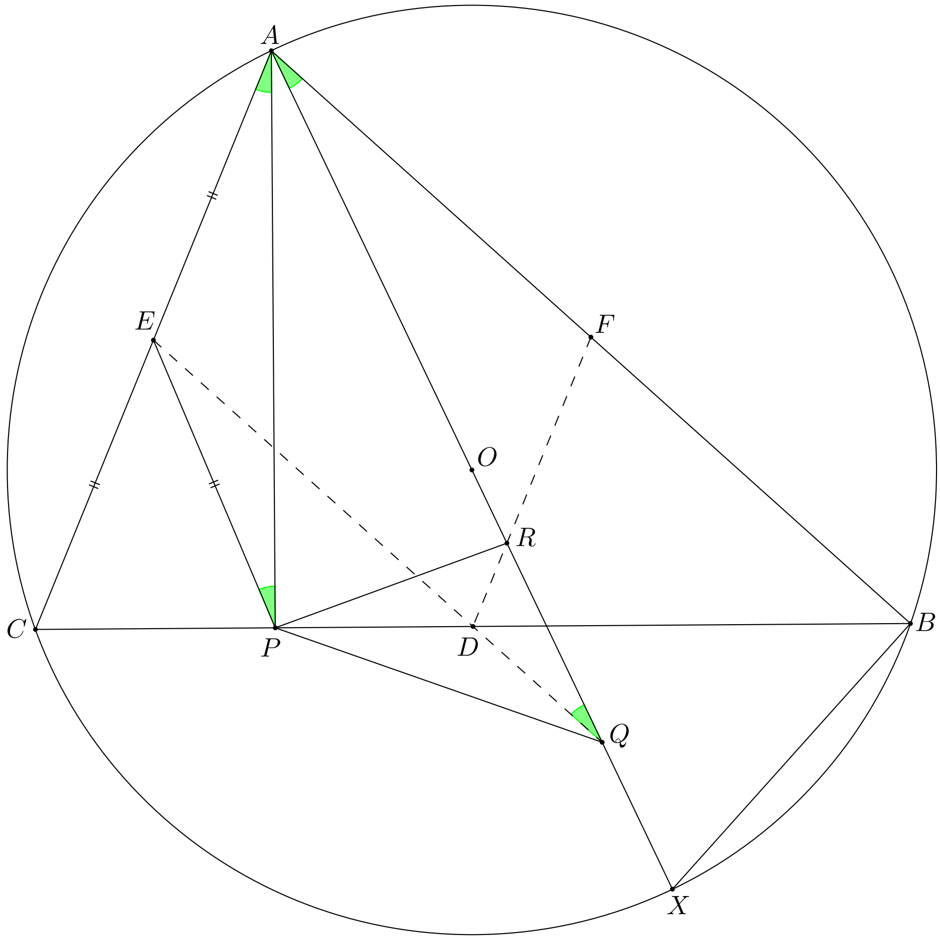
where $q_1, q_2, \ldots, q_m$ are the primes less than a_1. It follows that there are only finitely many choices for C.

4. Let ABC be an acute triangle with $AB > AC$. Let O be the circumcentre of triangle ABC and P be the foot of the altitude from A to BC. Denote the midpoints of the sides BC, CA and AB by D, E and F, respectively. The line AO intersects the lines DE and DF at Q and R, respectively.

   Prove that D is the incentre of triangle PQR.

Solution 1 (Sampson Wong)

Let AO intersect the circumcircle again at X. We will first show $\angle RQD = \angle DQP$ via angle chasing.



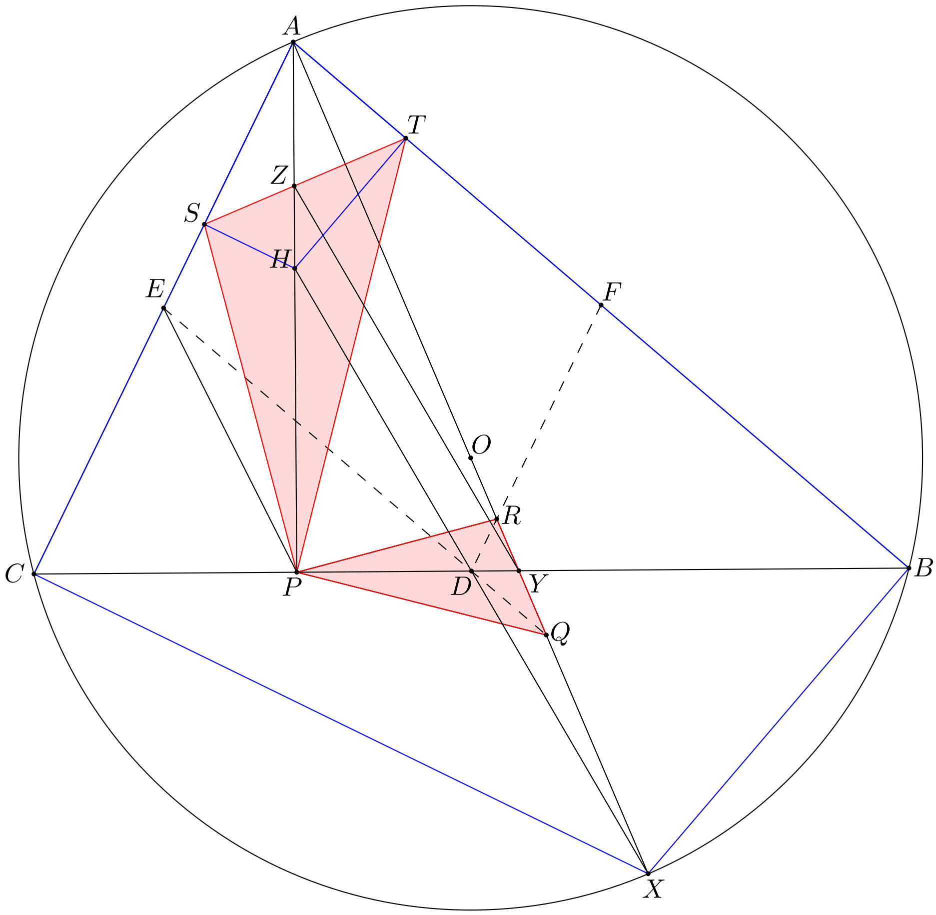
$$\begin{aligned}
\angle AQE &= \angle XAB && (AB \parallel DE) \\
&= 90^\circ - \angle BXA && (AX \text{ is a diameter}) \\
&= 90^\circ - \angle BCA && (ABXC \text{ is cyclic}) \\
&= \angle CAP && (AP \perp BC) \\
&= \angle APE && (AE = EP = EC)
\end{aligned}$$

So $AQPE$ is cyclic with $AE = EP$. Hence $\angle RQD = \angle DQP$ is proven.

By similar arguments, we also have $\angle DRQ = \angle PRD$. Therefore D is the incentre of $\triangle PQR$.

Solution 2 (Ivan Guo)

Let P, S and T be the feet of the altitudes as shown in the diagram below, so $\triangle PST$ is the orthic triangle of $\triangle ABC$. We would like to show that there is a spiral symmetry about P which sends $\triangle PST$ to $\triangle PRQ$ and H to D. This solves the problem since H can be easily proven to be the incentre of the orthic triangle. For example, by the cyclic quadrilaterals $ASHT$, $ACPT$ and $CPHS$, we have $\angle HST = \angle HAT = \angle PCH = \angle PSH$.



Extend AO to meet the circumcircle at X. Since CH, BX are both perpendicular to AB, while BH, CX are both perpendicular to AC, $CHBX$ is a parallelogram and D is the midpoint of HX.

Next, since $\angle SAH = 90^\circ - \angle BCA = 90^\circ - \angle BXA = \angle XAB$ and $\angle HSA = \angle ABX = 90^\circ$, we have $\triangle SAH \sim \triangle BAX$. Similarly, $\triangle TAH \sim \triangle CAX$. Therefore, $ASHT$ and $ABXC$ are similar quadrilaterals related by a "flip dilation", or a reflection composed with a dilation, about the angle bisector of $\angle BAC$. Since AX is the reflection of AH, BC is parallel to the reflection of ST, and $AH \perp BC$, we must have $AX \perp ST$.

Denote the intersection of AX and BC by Y, and the intersection of AH and ST by Z. Then,

$$ASHT \sim ABXC \implies AY/AX = AZ/AH \implies YZ \parallel XH \implies PD/PY = PH/PZ.$$

Consider the 90° spiral symmetry with centre P that sends ZH to YD. Since $DQ \perp HT$, $DR \perp HS$ and $RQ \perp ST$, the spiral symmetry must send $\triangle HST$ to $\triangle DRQ$. Therefore, we have constructed a spiral symmetry that sends $\triangle PST$ to $\triangle PRQ$ and H to D, as desired.

5. Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\big(x^{f(y)}\big) = f(x)^y$$

for all positive real numbers x and y.

Solution (Angelo Di Pasquale)

The only functions satisfying the given condition are $f(x) = 1$ and $f(x) = x$. Note that $f(x) = 1$ is certainly a solution. Assume henceforth that $f(a) \neq 1$ for some a.

If $f(y_1) = f(y_2)$, then substituting $(x, y) = (a, y_1)$ and (a, y_2) yields $y_1 = y_2$. Hence f is injective.

Observe that, by repeatedly applying the condition of the problem,

$$f\big(x^{f(y)f(z)}\big) = f\big((x^{f(y)})^{f(z)}\big) = \big(f(x^{f(y)})\big)^z = \big(f(x)^y\big)^z = f(x)^{yz} = f\big(x^{f(yz)}\big).$$

Since f is injective, this implies the multiplicative condition

$$f(yz) = f(y)f(z) \quad \text{for all } y, z \in \mathbb{R}^+.$$

Building upon this, we have

$$f\big(x^{f(y)+f(z)}\big) = f\big(x^{f(y)}x^{f(z)}\big) = f\big(x^{f(y)}\big)f\big(x^{f(z)}\big) = f(x)^y f(x)^z = f(x)^{y+z} = f\big(x^{f(y+z)}\big).$$

Since f is injective, this implies the additive condition

$$f(y + z) = f(y) + f(z) \quad \text{for all } y, z \in \mathbb{R}^+.$$

By induction, the additive condition implies that $f(nx) = nf(x)$ for all $n \in \mathbb{N}^+$ and $x \in \mathbb{R}^+$. Hence, for all $p, q \in \mathbb{N}^+$, $qf(p/q) = f(p) = pf(1)$, which implies that $f(x) = xf(1)$ for all $x \in \mathbb{Q}^+$. Putting this into the multiplicative condition yields $f(1) = 1$. So

$$f(x) = x \quad \text{for all } x \in \mathbb{Q}^+.$$

Again from the additive condition, if $a < b$, then

$$f(b) = f(b - a) + f(a) > f(a).$$

Therefore f is strictly increasing. This allows us to extend $f(x) = x$ from the positive rationals to the positive reals, as every real number can be approximated arbitrarily closely from both above and below by rationals.

Finally, both $f(x) = 1$ and $f(x) = x$ can be easily checked to be valid solutions.