AUSTRALIAN MATHS TRUST

ASC.

AMOC SENIOR CONTEST

DATE

Tuesday 17 August 2021

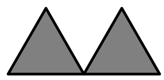
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Let $ABCD$ be a quadrilateral such that $\angle DAB = \angle CDA = 90^\circ$. Diagonals AC and BD meet at M. Let K be a point on side AD such that $\angle ABK = \angle DCK$.

   Prove that KM bisects $\angle BKC$.

2. For a positive integer n, let T_n be an equilateral triangle of side length n. The triangle T_n is divided into a triangular grid of unit triangles using lines parallel to the sides of T_n. (Each unit triangle is an equilateral triangle of side length 1.)

   A *saw-tooth* consists of two unit triangles joined at a vertex, producing a shape that is congruent to the following figure.

   A *saw-tooth tiling* of T_n is a placement of saw-teeth such that each saw-tooth exactly covers two unit triangles in the grid and each unit triangle in the grid is covered exactly once.

   For which values of n does T_n have a saw-tooth tiling?

3. Amy and Ben each have a list of 2021 positive integers. Is it possible for all three of the following conditions to hold at the same time?

   (i) All 4042 integers are different from each other.

   (ii) The sum of Amy's integers is equal to the sum of Ben's integers.

   (iii) The sum of the squares of Amy's integers is equal to the sum of the squares of Ben's integers.

4. Let M be the midpoint of side BC in triangle ABC. The tangent at B to the circle through A, B and M intersects the line AC at P. The circle through P, A and M intersects the line PB again at Q.

   Prove that the circle through Q, M and C is tangent to the line AC.

5. Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x^2 + xf(y)) = f(f(x))(x + y)$$
   for all positive real numbers x and y.