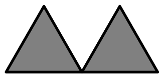
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Let $ABCD$ be a quadrilateral such that $\angle DAB = \angle CDA = 90°$. Diagonals AC and BD meet at M. Let K be a point on side AD such that $\angle ABK = \angle DCK$.

   Prove that KM bisects $\angle BKC$.

2. For a positive integer n, let T_n be an equilateral triangle of side length n. The triangle T_n is divided into a triangular grid of unit triangles using lines parallel to the sides of T_n. (Each unit triangle is an equilateral triangle of side length 1.)

   A *saw-tooth* consists of two unit triangles joined at a vertex, producing a shape that is congruent to the following figure.

   A *saw-tooth tiling* of T_n is a placement of saw-teeth such that each saw-tooth exactly covers two unit triangles in the grid and each unit triangle in the grid is covered exactly once.

   For which values of n does T_n have a saw-tooth tiling?

3. Amy and Ben each have a list of 2021 positive integers. Is it possible for all three of the following conditions to hold at the same time?

   (i) All 4042 integers are different from each other.

   (ii) The sum of Amy's integers is equal to the sum of Ben's integers.

   (iii) The sum of the squares of Amy's integers is equal to the sum of the squares of Ben's integers.

4. Let M be the midpoint of side BC in triangle ABC. The tangent at B to the circle through A, B and M intersects the line AC at P. The circle through P, A and M intersects the line PB again at Q.

   Prove that the circle through Q, M and C is tangent to the line AC.

5. Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x^2 + xf(y)) = f(f(x))(x + y)$$
   for all positive real numbers x and y.

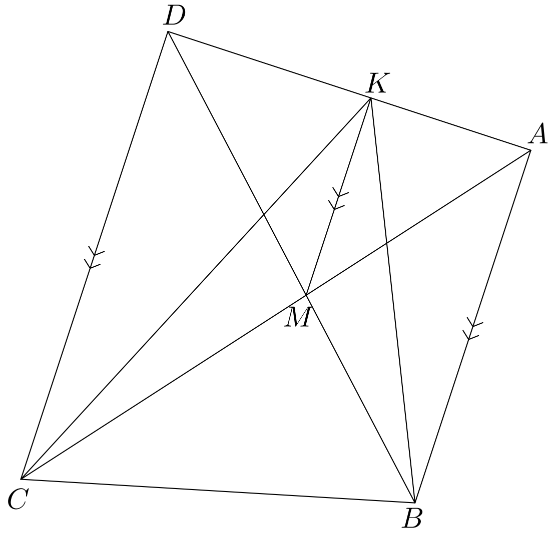
1. Let $ABCD$ be a quadrilateral such that $\angle DAB = \angle CDA = 90°$. Diagonals AC and BD meet at M. Let K be a point on side AD such that $\angle ABK = \angle DCK$.

   Prove that KM bisects $\angle BKC$.

Solution 1 (Oleksiy Yevdokimov)

Since $\angle DAB = \angle CDA = 90°$ and $\angle ABK = \angle DCK$, triangles CDK and BAK are similar, so we have $CD/AB = DK/KA$. Since CD and AB are parallel, triangles CDM and ABM are also similar, so we have $CD/AB = DM/MB$.

Hence $DK/KA = DM/MB$. Therefore CD, AB and KM are all parallel. Finally, $\angle MKC = \angle DCK = \angle ABK = \angle MKB$, as required.



Solution 2 (Angelo Di Pasquale)

Angle chasing yields $\angle BKA = \angle DKC$. So $\angle CKM = \angle MKB$ if and only if $KM \perp AD$. We use coordinates to establish this.

Let $K = (0, 0)$, $A = (0, a)$, $D = (0, d)$, $B = (b, a)$ and $C = (c, d)$. Since $\angle BKA = \angle DKC$, we have

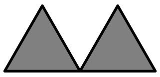
$$\text{grad}(KB) = -\text{grad}(KC) \quad \iff \quad a/b = -d/c \quad \iff \quad ac = -bd.$$

The equation of the line AC is given by $y - a = \frac{d-a}{c}x$. Its intersection with the x-axis is given by $-a = \frac{d-a}{c}x$, that is, $x = -\frac{ac}{d-a}$. The equation of the line BD is given by $y - d = \frac{a-d}{b}x$. Its intersection with the x-axis is given by $-d = \frac{a-d}{b}x$, that is, $x = -\frac{bd}{a-d}$.

Since $bd = -ac$, the lines AC and BD intersect the x-axis at the same point. So M is on the x-axis. Therefore $KM \perp AD$, so KM bisects $\angle BKC$ as desired.

2. For a positive integer n, let T_n be an equilateral triangle of side length n. The triangle T_n is divided into a triangular grid of unit triangles using lines parallel to the sides of T_n. (Each unit triangle is an equilateral triangle of side length 1.)

   A *saw-tooth* consists of two unit triangles joined at a vertex, producing a shape that is congruent to the following figure.

   A *saw-tooth tiling* of T_n is a placement of saw-teeth such that each saw-tooth exactly covers two unit triangles in the grid and each unit triangle in the grid is covered exactly once.

   For which values of n does T_n have a saw-tooth tiling?

Solution (Brenton Gray)

We will show that T_n can be tiled by saw-teeth if and only if n is a multiple of 4.

First of all, each unit triangle T_1 can either point *up* or *down*. In fact, if T_n is pointing up, it is made up of $1 + 2 + \cdots + n = n(n + 1)/2$ unit triangles that point up and $1 + 2 + \cdots + (n - 1) = n(n - 1)/2$ unit triangles that point down. In the example below, we see that T_3 is made up of six unit triangles that point up (black) and three that point down (white).

Since each saw-tooth consists of two unit triangles of the same orientation, in order to achieve a tiling, we need both $n(n + 1)/2$ and $n(n - 1)/2$ to be even. Now $n(n + 1)/2$ is even if and only if n is 0 or -1 mod 4, while $n(n - 1)/2$ is even if and only if n is 0 or 1 mod 4. Thus n must be a multiple of 4.

It suffices to provide a construction when $n = 4k$. This can be done by dissecting T_{4k} into copies of T_4 and tiling each T_4 as shown below.

3. Amy and Ben each have a list of 2021 positive integers. Is it possible for all three of the following conditions to hold at the same time?

   (i) All 4042 integers are different from each other.

   (ii) The sum of Amy's integers is equal to the sum of Ben's integers.

   (iii) The sum of the squares of Amy's integers is equal to the sum of the squares of Ben's integers.

Solution 1 (Angelo Di Pasquale)

Yes, it is possible for the two lists of positive integers to satisfy the conditions of the problem. We will prove this in the case where each list has n positive integers, where $n \geq 3$. The given problem corresponds to the case $n = 2021$.

Let A denote the set of Amy's numbers and let B denote the set of Ben's numbers. Suppose that $A = \{x+1, x+2, \ldots, x+n-2, x+n-1, x-N\}$ and $B = \{x-1, x-2, \ldots, x-(n-2), x-(n-1), x+N\}$, where $N = 1+2+\cdots+(n-1)$. If $x > N$, it is readily verified that A and B are disjoint and that

$$\sum_{i \in A} i = \sum_{i \in B} i = nx \quad \text{and} \quad \sum_{i \in A} i^2 = \sum_{i \in B} i^2 = nx^2 + N^2 + \sum_{i=1}^{n-1} i^2.$$

Solution 2 (Angelo Di Pasquale)

Yes, it is possible for the two lists of positive integers to satisfy the conditions of the problem. We will prove this in the case where each list has n positive integers, where $n \geq 6$. The given problem corresponds to the case $n = 2021$.

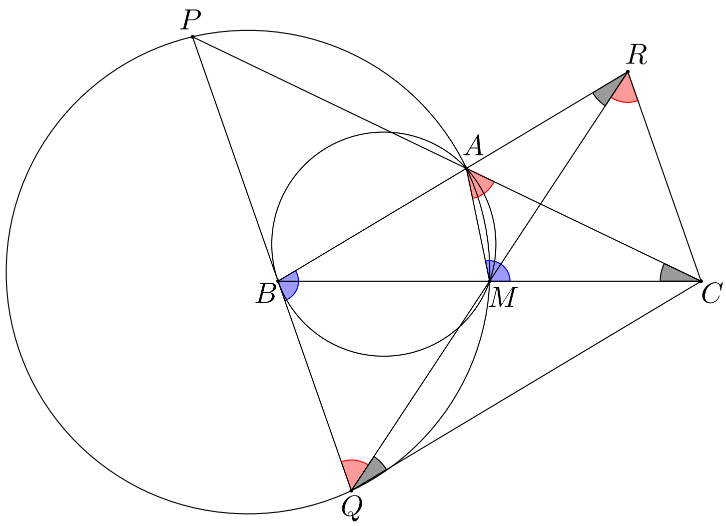
Let S be the set of positive integers for which it is possible. If $x, y \in S$, then $x + y \in S$, because if A_x, B_x are working sets for x, and A_y, B_y are working sets for y, then $A_x \cup cA_y$ and $B_x \cup cB_y$ are working sets for $x + y$, where c is chosen to make the smallest member of $cA_y \cup cB_y$ greater than the largest member of $A_x \cup B_x$.

Note that $A = \{1, 5, 6\}$ and $B = \{2, 3, 7\}$ show that $3 \in S$. Also $A = \{1, 4, 6, 7\}$ and $B = \{2, 3, 5, 8\}$ show that $4 \in S$. Since any $n \geq 6$ can be written in the form $3u + 4v$ for non-negative integers u and v, the result follows.

4. Let M be the midpoint of side BC in triangle ABC. The tangent at B to the circle through A, B and M intersects the line AC at P. The circle through P, A and M intersects the line PB again at Q.

   Prove that the circle through Q, M and C is tangent to the line AC.

Solution 1 (Angelo Di Pasquale)



Let $R = QM \cap AB$. From circle $PAMQ$ and the alternate segment theorem on circle ABM we have

$$\angle ACM = \angle AMB - \angle MAC = \angle RBP - \angle RQB = \angle BRM.$$

Hence $ARCM$ is cyclic and so

$$\angle RQP = \angle MAC = \angle MRC.$$

Hence $BQ \parallel RC$. Since M is the midpoint of BC it follows that $BRCQ$ is a parallelogram. Therefore

$$\angle ACM = \angle ARQ = \angle CQR.$$

Hence the circle through Q, M and C is tangent to the line AC.

Remark There is an alternate diagram where Q is between P and B. However that case can be taken care of by analogous arguments or using directed angles.

Solution 2 (Michelle Chen)

Let S be the reflection of A about M. Then, $ACSB$ is a parallelogram since $AM = MS$ and $BM = MC$. From this and circle $PAMQ$, we have

$$\angle MSB = \angle MAC = \angle MQP = \angle MQB.$$

Hence $BQSM$ is cyclic. Then, the alternate segment theorem on circle ABM gives

$$\angle ASQ = \angle MSQ = 180° - \angle QBM = \angle 180° - \angle BAM = 180° - \angle BAS.$$

Hence $AB \parallel SQ$. Since $ACSB$ is a parallelogram, we also have $AB \parallel CS$. It follows that C, S, and Q are collinear. Therefore

$$\angle CQM = \angle SQM = \angle SBM = \angle ACM.$$

Hence the circle through Q, M and C is tangent to the line AC.

Solution 3 (Thanom Shaw)

We proceed via reverse reconstruction. Define Q' to be the intersection of PB and the line through C parallel to AB. We prove that Q' is in fact Q by showing that $PAMQ'$ is cyclic.

Let AM meet the line through C parallel to AB at D. Since $AB \parallel CD$ and PQ' is tangent to the circumcircle of ABM,

$$\angle DQ'B = \angle ABP = \angle AMB.$$

Hence $BMDQ'$ is cyclic.

Since M is the midpoint of BC and $AB \parallel CD$, $\triangle AMB \equiv \triangle DMC$ and so $ABDC$ is a parallelogram and $BD \parallel PC$.

Hence with $BMDQ'$ cyclic and $BD \parallel PC$,

$$\angle DMQ' = \angle DBQ' = \angle APB.$$

Therefore $PAMQ'$ is cyclic and $Q' = Q$.

Finally, since $BMDQ' = BMDQ$ is cyclic and $BD \parallel PC$,

$$\angle DQM = \angle DBM = \angle MCA$$

and hence the circle through Q, M, and C is tangent to the line AC.

5. Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x^2 + xf(y)) = f(f(x))(x + y)$$

for all positive real numbers x and y.

Solution 1 (William Steinberg)

Answer: $f(x) = x$.

Since $f(f(x)) > 0$, by varying y, we see that f is injective. Consider $x \in (0, 1)$ and substitute $y = 1-x$. Applying injectivity we get $x^2+xf(1-x) = f(x)$. Further substituting $1 - x$ into x we get the simultaneous equations

$$\begin{aligned} x^2 + xf(1-x) &= f(x) \\ (1-x)^2 + (1-x)f(x) &= f(1-x). \end{aligned}$$

We can solve this to obtain

$$f(x) = \frac{x(1-x)^2 + x^2}{1 - x(1-x)} = x$$

for all $x \in (0, 1)$.

We now prove by induction on n that $f(x) = x$ for $x \in (0, n)$. Suppose this is true for $n = k$, then substitute $x, y \in (0, k)$. This gives

$$f(x(x + y)) = f(x^2 + xf(y)) = f(f(x))(x + y) = x(x + y).$$

However $x(x + y)$ covers all values in $(0, k + 1) \subseteq (0, 2k^2)$ as $x, y \in (0, k)$ vary. This completes the induction. Thus $f(x) = x$ for all $x > 0$.

Finally, the solution $f(x) = x$ can be easily verified to satisfy the problem condition.

Solution 2 (Alice Devillers)

After reaching $f(x) = x$ for all $0 < x < 1$, take $x = \frac{\sqrt{17}-1}{4}, y = \frac{1}{2}$ so that $x^2 + \frac{x}{2} = 1$ and $0 < x, y < 1$. We get

$$\begin{aligned} f(1) = f\left(\left(\frac{\sqrt{17}-1}{4}\right)^2 + \frac{1}{2}\left(\frac{\sqrt{17}-1}{4}\right)\right) &= f\left(f\left(\frac{\sqrt{17}-1}{4}\right)\right)\left(\frac{\sqrt{17}-1}{4} + \frac{1}{2}\right) \\ &= \frac{\sqrt{17}-1}{4} \cdot \frac{\sqrt{17}+1}{4} = 1. \end{aligned}$$

Thus $f(x) = x$ for all $0 < x \le 1$. Now taking $x = 1$, we have, for all y,

$$f(1 + f(y)) = 1 + y.$$

In particular, $f(y) = y$ implies $f(y + 1) = y + 1$. Since $f(x) = x$ holds for all $0 < x \le 1$, it follows by induction that $f(x) = x$ holds for all $n < x \le n + 1$ where n is any positive integer. Thus $f(x) = x$ for all $x > 0$.

Solution 3 (Norman Do)

This is an alternative way to finish once it is established that f is injective and that $f(x) = x$ whenever $0 < x < 1$.

For any fixed value of y, choose x sufficiently small so that $0 < x < 1$ and $x^2 + xf(y) < 1$. Putting this value of x into the given functional equation and using $f(z) = z$ whenever $0 < z < 1$ yields

$$x^2 + xf(y) = x(x + y)$$

which implies $f(y) = y$. Thus $f(y) = y$ for all $y > 0$.

Statistics

Score Distribution/Problem

Number of Students/Score	Problem Number				
	1	**2**	**3**	**4**	**5**
0	19	4	90	98	80
1	11	7	4	4	18
2	15	15	8	9	8
3	0	15	1	0	6
4	2	2	1	0	3
5	0	5	2	2	1
6	0	9	2	0	1
7	90	80	29	24	20
Mean	**5.0**	**5.3**	**1.8**	**1.5**	**1.6**