Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all triples of positive integers (a, b, n) such that

$$2^a + b^2 = n! + 14.$$

(Note that $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

2. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers, where $n \geq 2$.

Prove that

$$x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_{n-1}x_n \leq \frac{(x_1 + x_2 + \cdots + x_n)^2}{4}.$$

(Note that each term on the left-hand side is of the form x_ix_{i+1} for $i = 1, 2, \ldots, n-1$.)

3. Let ABC be a triangle with incentre I. Let $\mathcal{K}$ be the circle that passes through I and is tangent to the circumcircle of ABC at A. Suppose that $\mathcal{K}$ intersects AB and AC again at P and Q, respectively.

Prove that the angle bisectors of $\angle BPQ$ and $\angle PQC$ intersect on BC.

4. For a real number x, let $\lceil x \rceil$ be the smallest integer greater than or equal to x.

Find all nonempty sets S of positive integers such that whenever a and b (not necessarily distinct) are in S, then both ab and $\left\lceil \frac{a}{b} \right\rceil$ are in S.

5. Hugo and Maryna are playing a game with n buckets and an infinite pile of stones, where n is a positive integer. Initially, all buckets are empty.

Hugo and Maryna alternate their turns, with Hugo going first. During Hugo's turn, he picks up two stones from the pile and either puts them into one bucket of his choice or into two separate buckets of his choice. During Maryna's turn, she chooses one of the n buckets and empties it back onto the pile.

Hugo wins if, at any point, one of the buckets contains at least 50 stones.

For which values of n does Hugo have a winning strategy?

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