Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all triples of positive integers (a, b, n) such that

$$2^a + b^2 = n! + 14.$$

(Note that $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

2. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers, where $n \geq 2$.

Prove that

$$x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_{n-1}x_n \leq \frac{(x_1 + x_2 + \cdots + x_n)^2}{4}.$$

(Note that each term on the left-hand side is of the form x_ix_{i+1} for $i = 1, 2, \ldots, n-1$.)

3. Let ABC be a triangle with incentre I. Let $\mathcal{K}$ be the circle that passes through I and is tangent to the circumcircle of ABC at A. Suppose that $\mathcal{K}$ intersects AB and AC again at P and Q, respectively.

Prove that the angle bisectors of $\angle BPQ$ and $\angle PQC$ intersect on BC.

4. For a real number x, let $\lceil x \rceil$ be the smallest integer greater than or equal to x.

Find all nonempty sets S of positive integers such that whenever a and b (not necessarily distinct) are in S, then both ab and $\left\lceil \frac{a}{b} \right\rceil$ are in S.

5. Hugo and Maryna are playing a game with n buckets and an infinite pile of stones, where n is a positive integer. Initially, all buckets are empty.

Hugo and Maryna alternate their turns, with Hugo going first. During Hugo's turn, he picks up two stones from the pile and either puts them into one bucket of his choice or into two separate buckets of his choice. During Maryna's turn, she chooses one of the n buckets and empties it back onto the pile.

Hugo wins if, at any point, one of the buckets contains at least 50 stones.

For which values of n does Hugo have a winning strategy?

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1. Find all triples of positive integers (a, b, n) such that

$$2^a + b^2 = n! + 14.$$

(Note that $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

Solution (Mike Clapper)

Answer: $(a, b, n) = (2, 4, 3), (4, 2, 3)$ and $(1, 6, 4)$.

First consider the case of $n \geq 4$, we have $2^a + b^2 = n! + 14 \equiv 2 \pmod 4$. If $a \geq 2$, then $2^a + b^2 \equiv 0$ or $1 \pmod 4$, which is a contradiction. So $a = 1$ and the required equation becomes $b^2 = n! + 12$. If $n \geq 5$, then $b^2 = n! + 12 \equiv 2 \pmod 5$, but this is impossible since $b^2 \equiv 0, 1$ or $4 \pmod 5$. So the only possibility here is $n = 4$, which leads to the solution $(a, b, n) = (1, 6, 4)$.

For $n \leq 3$ the problem reduces to a finite case check.

- For $n = 3$, we have $2^a + b^2 = 20$. It suffices to only check $a \leq 4$, which yields the solutions $(a, b, n) = (2, 4, 3)$ and $(4, 2, 3)$.
- For $n = 2$ or 1, we have $2^a + b^2 = 16$ or 15. It suffices to only check $a \leq 3$, which yields no solutions.

Remark. Other modulo arguments can also be used to narrow the problem down to a finite case check. For example: $b^2 = n! + 12$ and $n \geq 6$ can be ruled out by using mod 9 instead of mod 5; the case $n \geq 7$ can be ruled out by using mod 7.

2. Let $x_1, x_2, \dots, x_n$ be non-negative real numbers, where $n \geq 2$.

Prove that

$$x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_{n-1}x_n \leq \frac{(x_1 + x_2 + \cdots + x_n)^2}{4}.$$

(Note that each term on the left-hand side is of the form $x_i x_{i+1}$ for $i = 1, 2, \dots, n-1$.)

Solution 1 (Kevin McAvaney)

The required inequality is equivalent to

$$\begin{aligned} 4(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n) &\leq \sum_{i=1}^{n} \big(x_i^2 + 2x_i(x_{i+1} + x_{i+2} + \cdots + x_n)\big) \\ \iff \qquad 0 &\leq \sum_{i=1}^{n} \big(x_i^2 - 2x_ix_{i+1} + 2x_i(x_{i+2} + x_{i+3} + \cdots + x_n)\big). \end{aligned}$$

This is true by the follow inequalities:

$$\begin{aligned} 0 &\leq (x_1 - x_2 + x_3 - x_4 + \cdots + (-1)^{n-1}x_n)^2 \\ &= \sum_{i=1}^{n} \big(x_i^2 + 2x_i(-x_{i+1} + x_{i+2} - x_{i+3} + \cdots + (-1)^{i+n}x_n)\big) \\ &= \sum_{i=1}^{n} \big(x_i^2 - 2x_ix_{i+1} + 2x_i(x_{i+2} - x_{i+3} + \cdots + (-1)^{i+n}x_n)\big) \\ &\leq \sum_{i=1}^{n} \big(x_i^2 - 2x_ix_{i+1} + 2x_i(x_{i+2} + x_{i+3} + \cdots + x_n)\big). \end{aligned}$$

Hence the original inequality is valid.

Solution 2 (Ivan Guo)

We proceed by induction. The base case of $n = 2$ follows from

$$x_1x_2 \leq \frac{(x_1 + x_2)^2}{4} \qquad \iff \qquad 0 \leq (x_1 - x_2)^2.$$

Next assume the result holds for $n = k$ and consider $n = k+1$. Without loss of generality, we may assume that $x_2 \geq x_k$ (if not we can reverse the sequence). Then we have

$$x_1x_2 + x_2x_3 + \cdots + x_kx_{k+1} \leq (x_1 + x_{k+1})x_2 + x_2x_3 + \cdots + x_{k-1}x_k.$$

Applying the $n = k$ case to the real numbers $(x_1 + x_{k+1}), x_2, \dots, x_k$ yields

$$(x_1 + x_{k+1})x_2 + x_2x_3 + \cdots + x_{k-1}x_k \leq \frac{(x_1 + x_{k+1} + x_2 + \cdots + x_k)^2}{4},$$

completing the induction.

Solution 3 (Norman Do)

Use the AM-GM inequality as follows.

$$\sqrt{(x_1 + x_3 + \cdots) \times (x_2 + x_4 + \cdots)} \leq \frac{(x_1 + x_3 + \cdots) + (x_2 + x_4 + \cdots)}{2}$$

After squaring both sides, we obtain

$$(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n) + (\text{extra terms}) \leq \frac{(x_1 + x_2 + \cdots + x_n)^2}{4}.$$

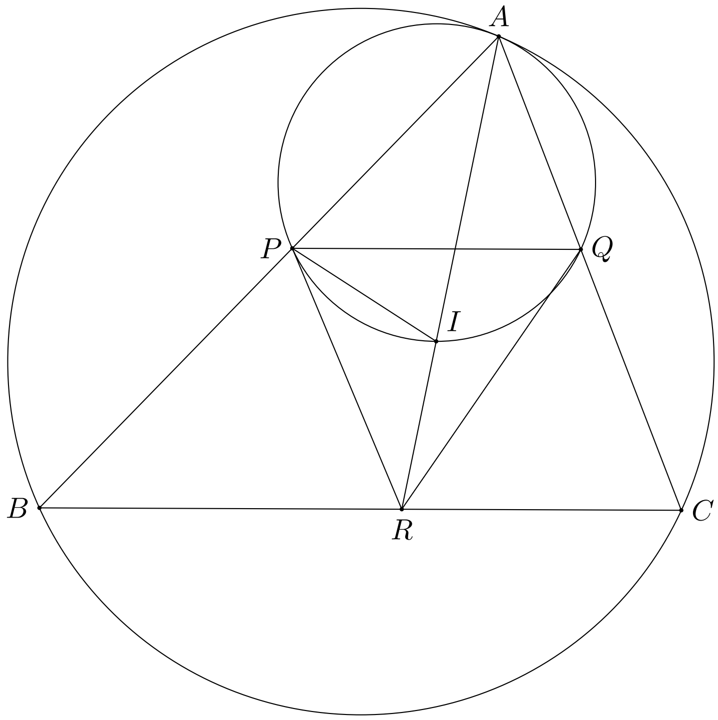
It then suffices to observe that the extra terms are sums of products of the original non-negative numbers $x_1, x_2, \ldots, x_n$.

3. Let ABC be a triangle with incentre I. Let $\mathcal{K}$ be the circle that passes through I and is tangent to the circumcircle of ABC at A. Suppose that $\mathcal{K}$ intersects AB and AC again at P and Q, respectively.

   Prove that the angle bisectors of $\angle BPQ$ and $\angle PQC$ intersect on BC.

Solution 1 (Sampson Wong)

Extend AI to intersect BC at R.



It suffices to show that $\angle BPR = \angle RPQ$ (and similarly $\angle PQR = \angle RQC$). By dilating the small circle to the larger one about A, we see that $PQ \parallel BC$. So it suffices to show that $\angle BPR = \angle PRB = \angle RPQ$. Hence it is enough to show that $BP = BR$.

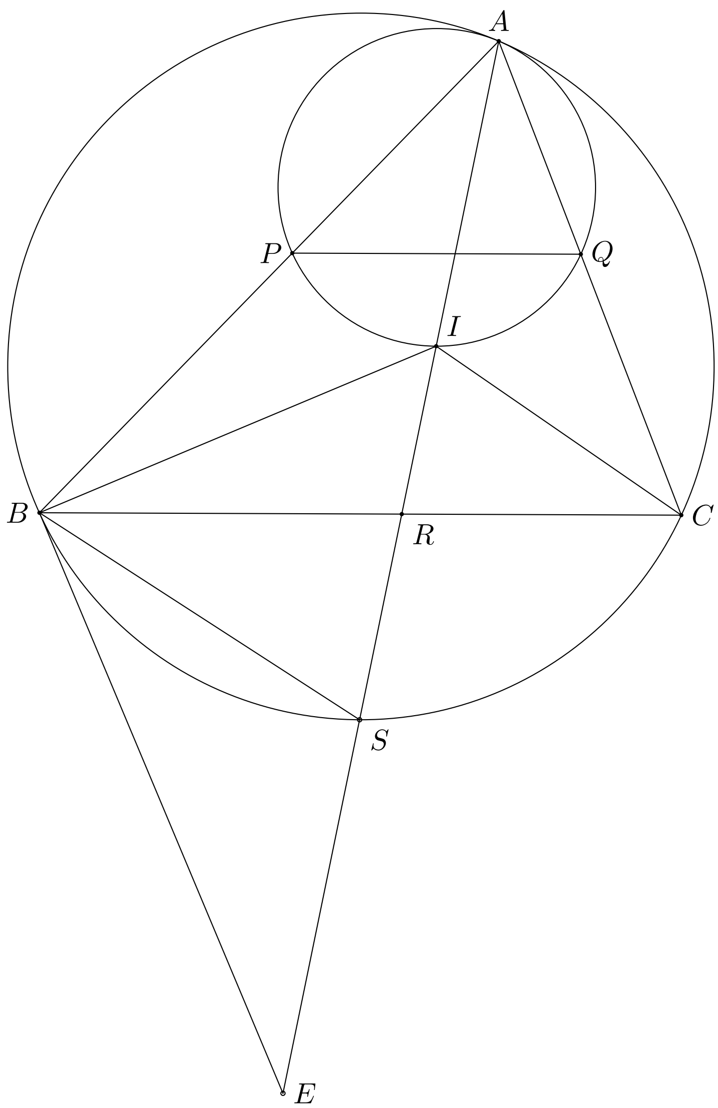
Now,

$$\begin{aligned} \angle BPI &= \angle PAI + \angle AIP \\ &= \angle IAQ + \angle AQP \qquad (APIQ \text{ is cyclic, } I \text{ is the incentre}) \\ &= \angle RAC + \angle ACR \qquad (PQ \parallel BC) \\ &= \angle IRB. \end{aligned}$$

In $\triangle BPI$ and $\triangle IRB$, we have $\angle BPI = \angle IRB$, $\angle IBP = \angle RBI$, and IB is common. Therefore $\triangle BPI \cong \triangle IRB$, and $BP = BR$ as required.

Solution 2 (Ivan Guo)

Let AI meet BC at R and the circumcircle at S. Then since $\angle BAS = \angle RAC$ (angle bisector) and $\angle ASB = \angle ACB$ ($ABSC$ is cyclic), we must have $\triangle BAS \sim \triangle RAC$ and hence $AB \times AC = AS \times AR$.



Next, let E be the excentre of $\triangle ABC$ opposite A, so $\angle EBI = 90^\circ$ (internal and external angle bisectors are perpendicular). Then $\angle BAE = \angle IAC$ (angle bisector) and

$$\begin{aligned}\angle EBA &= \angle EBI + \angle IBA \\ &= 90^\circ + \frac{\angle CBA}{2} \\ &= 90^\circ + \frac{180^\circ - \angle BAC - \angle ACB}{2} \\ &= 180^\circ - \angle IAC - \angle ACI \\ &= \angle CIA.\end{aligned}$$

Hence $\triangle BAE \sim \triangle IAC$ and $AB \times AC = AI \times AE$.

Combining the above we get

$$AS \times AR = AB \times AC = AI \times AE \quad \Longrightarrow \quad AI/AS = AR/AE.$$

Hence the dilation that sends $ABSC$ to $APIQ$ also sends E to R. Thus R is the excentre of $\triangle APQ$ opposite A, as required.

Solution 3 (Sally Tsang)

We will use reverse reconstruction. Let the angles of $\triangle ABC$ at A, B and C be $2\alpha, 2\beta$ and 2γ, respectively. Let the bisectors of $\angle BPQ$ and $\angle PQC$ meet at R. Let the line PR intersect BC at R', and intersect BI at X.

By the alternate segment theorem at the common tangent (or by dilation), $\angle APQ = \angle ABC = 2\beta$. Thus $\angle BPR = 90° - \beta$ and so $\angle PXB = 90° = \angle R'XB$. Hence $\triangle PXB \cong \triangle R'XB$ (AAS) and so $PX = XR'$.

Since $APIQ$ is cyclic, $\angle PIQ = 180° - 2\alpha$. Also, since AI is the angle bisector of $\angle PAQ$, we have $IP = IQ$. Angle-chasing in $\triangle PRQ$ yields $\angle PRQ = \beta + \gamma = 90° - \alpha$. Hence I is the circumcentre of $\triangle PRQ$. Then IX must be the perpendicular bisector of PR. But we know that $PX = XR'$, so $R = R'$, which is on BC.

4. For a real number x, let $\lceil x \rceil$ be the smallest integer greater than or equal to x.

Find all nonempty sets S of positive integers such that whenever a and b (not necessarily distinct) are in S, then both ab and $\lceil \frac{a}{b} \rceil$ are in S.

Solution 1 (Angelo Di Pasquale)

Answer: $S = \mathbb{N}^+$ or $S = S_r = \{1, r, r^2, r^3, \ldots\}$ for some positive integer r.

Taking $b = a$ shows that $1 \in S$. Note that $S = S_1 = \{1\}$ works. So suppose that $|S| > 1$.

Let r denote the smallest element of S that is bigger than 1. It follows that $S_r = \{1, r, r^2, r^3, \ldots\} \subseteq S$. Note that $S = S_r$ also works. So from here on suppose that $S \neq S_r$.

Let s be the smallest element of S not in S_r. Then $r^{n-1} < s < r^n$ for some positive integer n.

Consider $t = \lceil \frac{s}{r^{n-1}} \rceil \in S$. Clearly $1 < t \leq r$. By the minimality of r, we have $t = r$. Therefore

$$r^{n-1}(r-1) < s < r^n.$$

Consider $u = \lceil \frac{r^n}{s} \rceil \in S$. We have

$$1 < u \leq \left\lceil \frac{r^n}{r^{n-1}(r-1)} \right\rceil = \left\lceil \frac{r}{r-1} \right\rceil = 2.$$

Thus $u = 2 \in S$. So $r = 2$, and we have $2^{n-1} < s < 2^n$ for some positive integer n. This implies $n \geq 2$.

If $2^{n-1} < s \leq 2^n - 2$, then $2^{n-2} < \lceil \frac{s}{2} \rceil < 2^{n-1}$ yields a smaller member of S not in S_2, which contradicts the minimality of s. Hence $s = 2^n - 1$.

Here is a little lemma: If $k, k+1 \in S$ and $k \geq 2$ then also $k+3 \in S$. This is because $k, k+1 \in S$ implies $(k+1)^2 \in S$ which implies $\left\lceil \frac{(k+1)^2}{k} \right\rceil = \lceil k + 2 + \frac{1}{k} \rceil = k + 3 \in S$.

Applying the little lemma to $s, s+1 \in S$, we have $2^n + 2 \in S$, and so $\lceil \frac{2^n+2}{2} \rceil = 2^{n-1} + 1 \in S$. But this contradicts the minimality of s unless $s = 3$. Thus $3 \in S$. But now since $1, 2, 3, 4 \in S$, the little lemma inductively implies that all positive integers are in S.

Solution 2 (Angelo Di Pasquale)

Again note that $1 \in S$ and suppose that $|S| > 1$.

Let r be the smallest member of S that is greater than 1. Let $s > 1$ be any other member of S. Then also $\lceil \frac{s^m}{r^n} \rceil \in S$ for any $m, n \in \mathbb{Z}$. Let

$$F(m,n) = \log_r \left(\frac{s^m}{r^n} \right) = m \log_r s - n.$$

If $\log_r s$ is irrational, then the values of $F(m,n)$ form a dense subset of the real number line. Hence s^m / r^n form a dense subset of the positive reals. This implies that $S = \mathbb{N}^+$.

If $\log_r s$ is rational then $r^u = s^v$ for some relatively prime positive integers u, v. By examining their prime factorisations, we must have $r = t^v$ and $s = t^u$ for some integer $t > 1$. By Bézout's lemma, $\gcd(u,v)$ can be written as an integer linear combination of u and v, which implies $t^{\gcd(u,v)} = t \in S$. By the minimality of r it follows that $t = r$, and so s is a power of r. Since this is true for every $s \in S$, it follows that $S = \{1, r, r^2, r^3, \ldots\}$.

5. Hugo and Maryna are playing a game with n buckets and an infinite pile of stones, where n is a positive integer. Initially, all buckets are empty.

   Hugo and Maryna alternate their turns, with Hugo going first. During Hugo's turn, he picks up two stones from the pile and either puts them into one bucket of his choice or into two separate buckets of his choice. During Maryna's turn, she chooses one of the n buckets and empties it back onto the pile.

   Hugo wins if, at any point, one of the buckets contains at least 50 stones.

   For which values of n does Hugo have a winning strategy?

Solution (William Steinberg)

Answer: $n \geq 2^{47} + 1$.

We assign each bucket a rating of 2^i if it contains i stones where $i > 0$, and a rating of 0 if it is empty. We claim Maryna can ensure the sum of ratings stays strictly less than $2n$ after each of her moves.

- If Hugo puts 2 stones into the same bucket, then Maryna can empty this bucket, thus maintaining the sum below $2n$.
- If Hugo chooses two buckets with ratings a and b with $a \geq b > 0$, then the new ratings are $2a$ and $2b$. Maryna can empty the bucket with rating $2a$. Since $2b \leq a + b$, the sum stays below $2n$.
- If $a > b = 0$, then the new ratings are $2a$ and 2. Again Maryna can empty the bucket with rating $2a$. Since $2 \leq a$, the sum stays below $2n$.
- If $a = b = 0$, then Hugo's move increases the sum by 4.
  - If any bucket has at least 2 stones, Maryna can empty this bucket, decreasing the sum by at least 4.
  - If no bucket has more than 1 stone, Maryna can empty any bucket, and the sum of ratings will be below $2n$.

Thus the claim is proven.

For $n \leq 2^{47}$ Maryna can ensure the sum of rating remains less than 2^{48} after each of her turns. So there is no bucket with 48 stones or more after her turn and thus no bucket can ever reaches 50 stones.

We now show that for $n = 2^{47}+1$ Hugo can in fact win. Initially Hugo always chooses two empty buckets. Since Hugo chooses two buckets each turn and Maryna can only empty one, eventually after one of Hugo's turns every bucket will have 1 stone. After Maryna's next turn there will be exactly $n - 1 = 2^{47}$ buckets with 1 stone.

We now prove a lemma. If there are $2k$ buckets with r stones each before Hugo's turn, then Hugo can choose buckets in such a way so that after k rounds there are k buckets with $r+1$ stones each. To do this, he pairs up the $2k$ buckets and each turn he chooses two buckets from one pair, until all $2k$ are chosen after k rounds. In the meantime, Maryna can only empty at most k of these $2k$ buckets, so at least k buckets will survive with $r+1$ stones each. So the lemma is proven.

Hugo repeatedly applies this lemma until there is at least 1 bucket with 48 stones at the start of his turn. At this point he can win by simply putting 2 more stones into this bucket.

Statistics

Score Distribution/Problem

Mark/Problem	Q1	Q2	Q3	Q4	Q5
0	5	51	63	74	55
1	1	25	12	23	31
2	3	5	9	7	31
3	4	2	4	7	3
4	4	6	4	1	1
5	4	0	0	4	1
6	21	4	1	5	1
7	88	37	37	9	7
Average	**6.1**	**2.7**	**2.5**	**1.3**	**1.3**