MATHEMATICS OLYMPIAD TRAINING SESSIONS

2023 Senior Mathematics Contest Problems

1. For a string of As and Bs, define the substring replacement operation: $AB \longrightarrow BBAA$. Starting from any string of As and Bs, is it always possible to perform a sequence of such replacement operations to obtain a string where all Bs are to the left of all the As.

2. A positive integer m is given to Alice and Bob. Alice and Bob play a game:

   Alice goes first, writing a non-zero digit on the board.
   Then Bob and Alice alternate,
   appending a digit to the front or back of the current number on the board,
   except that a digit appended at the front of the number must be non-zero.

   Bob wins if at any time the number on the board is divisible by m.

   (i) Find the least m such that Alice can prevent Bob from winning.

   (ii) Same problem as (i), except Alice can write any $n \in \mathbb{N}$ at the start.

3. Points A, B, C, D lie on sides EF, FG, GH, HE, respectively, of a parallelogram $EFGH$. Also $AC \perp EF$, $BD \perp FG$, $ABCD$ is cyclic, and
   Q is the point on AC such that $FQ \parallel BC$.
   Prove that $EQ \parallel DC$.

4. Counters are placed, one at a time, in the unit squares of an $n \times n$ grid such that:

   (i) a counter can only be placed in an empty unit square,

   (ii) the first counter can be placed in any unit square,

   (iii) each subsequent counter can only be placed in a unit square S if

   the sum of the number of counters already in the same row as S and
   the number of counters already in the same column as S is odd.

   For each $n \geqslant 2$, find the smallest possible number of empty unit squares remaining after a sequence of such counter placements.

5. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be (not necessarily positive) real numbers, where $n \geqslant 2$.
   Let K, L be the maximum and minimum, respectively, of $b_1, b_2, \ldots, b_n$.
   Prove that $\sum_{i<j} a_i a_j |b_i - b_j| \leqslant \frac{1}{2}(K - L)(a_1 + a_2 + \cdots + a_n)^2$,
   where $\sum_{i<j}$ denotes the summing over all pairs (i, j) such that $1 \leqslant i < j \leqslant n$.