MATHEMATICS OLYMPIAD TRAINING SESSIONS

2023 Senior Mathematics Contest Problems with Some Solutions

1. For a string of As and Bs, define the substring replacement operation: $AB \longrightarrow BBAA$. Starting from any string of As and Bs, is it always possible to perform a sequence of such replacement operations to obtain a string where all Bs are to the left of all the As.

   Solution. Answer: No, it is not always possible after a sequence of replacement operations to obtain a string where all Bs are to left of all the As.

   If the initial string contains a substring AAB or ABB, the string always contains a substring AAB or ABB, in which case there is always at least one A to the left of a B.

   Proof. We suppose at some stage the string contains a substring AAB or ABB.

   Case 1: The string before application of the replacement operation contains AAB. After the replacement operation, either

   AAB is unchanged; then: final string contains AAB,

   or

   $AAB \longrightarrow ABBAA$; then: final string contains ABB.

   Case 2: The string before application of the replacement operation contains ABB. After the replacement operation, either

   ABB is unchanged. Then: final string contains ABB,

   or

   $ABB \longrightarrow BBAAB$. Then: final string contains AAB.

   So, in both cases, after each replacement operation, the string contains AAB or ABB. □

2. A positive integer m is given to Alice and Bob. Alice and Bob play a game:

   Alice goes first, writing a non-zero digit on the board.
   Then Bob and Alice alternate,
   appending a digit to the front or back of the current number on the board,
   except that a digit appended at the front of the number must be non-zero.

   Bob wins if at any time the number on the board is divisible by m.

   (i) Find the least m such that Alice can prevent Bob from winning.

   (ii) Same problem as (i), except Alice can write any $n \in \mathbb{N}$ at the start.

Solution. Answers: (i) $m = 12$. (ii) $m = 11$.

Proof. We either explain why, for a given m, Alice cannot prevent Bob from winning, or provide Alice's prevention strategy.

Case 1: $m \leqslant 10$ (for both (i) and (ii)).

After Alice writes starting number X, the possible numbers Bob can form:

$$\overline{Xd}$$

by appending $d \in \{0, 1, \dots, 9\}$, are 10 consecutive integer possibilities, (at least) one of which is divisible by $m \in \{1, 2, \dots, 10\}$.

Bob makes the appropriate choice and wins.

Case 2: $m = 11$.

For (i): After Alice writes $d \in \{1, 2, \dots, 9\}$,

Bob appends d obtaining: $\overline{dd}$ which is divisible by 11 and wins.

For (ii): Here Alice can prevent Bob from winning.

Alice always leaves a number $X \equiv -1 \pmod{11}$,

with an odd number of digits, initially choosing 120.

If Bob responds by appending $d \neq 0$ at front of X,

where X has k digits then

$$\begin{aligned}\overline{dX} &\equiv d \cdot 10^k + -1 \pmod{11}\\ &\equiv d \cdot (-1)^k + -1 \pmod{11}\\ &\equiv (d+1) \cdot -1 \pmod{11}, \text{ where } d+1 \in \{2, \dots, 10\}\\ &\not\equiv 0 \pmod{11}\end{aligned}$$

Alice responds by also appending $d \neq 0$ at front of X, so that

$$\overline{ddX} \equiv -1 \pmod{11}, \text{ since } 11 \,\big|\, \overline{dd} \cdot 10^k,$$

and $\overline{ddX}$ again has an odd number of digits.

Otherwise, if Bob responds by appending $d \in \{0, 1, \dots, 9\}$ at back of X then

$$\begin{aligned}\overline{Xd} &\equiv 10 \cdot -1 + d \pmod{11}\\ &\equiv 1 + d \pmod{11}, \text{ where } d+1 \in \{1, 2, \dots, 10\}\\ &\not\equiv 0 \pmod{11}\end{aligned}$$

Alice responds by also appending d at back of X, so that

$$\begin{aligned}\overline{Xdd} &\equiv 100 \cdot -1 + d \cdot 11 \pmod{11}\\ &\equiv 1 \cdot -1 + 0 \pmod{11}\\ &\equiv -1 \pmod{11}\end{aligned}$$

and $\overline{Xdd}$ again has an odd number of digits.

Case 3: $m = 12$ (for (i)).

Here Alice seeing X, can prevent Bob from winning,

by appending an appropriate $d \in \{1, 3, 5\}$ to the back of X.

Now, divisibility by $12 = 2^2 \cdot 3$, is equivalent to divisibility by both 4 and 3.

Firstly, since Alice forms an odd number $\overline{Xd}$,

Bob must append an even digit c after d, (since $n \vdots 12 \implies n \vdots 2$).

But for $\overline{Xdc}$ to be divisible by 4, we show Bob must append $c \in \{2, 6\}$:

$$\begin{aligned}\overline{Xdc} &\equiv \overline{dc} \pmod 4, \text{ since } X \cdot 100 \vdots 4\\ &\equiv \overline{1c} \pmod 4, \text{ since } 20 \vdots 4 \implies 50 \equiv 30 \equiv 10 \pmod 4\\ &\equiv 0 \pmod 4 \iff c \in \{2, 6\}.\end{aligned}$$

The choice d that Alice makes of 1, 3, 5 is to prevent Bob being able to make $\overline{Xdc}$ divisible by 3:

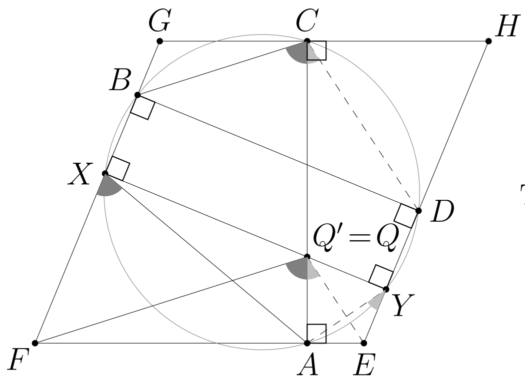
X mod 3	Alice appends d	Bob appends c	$\overline{Xdc}$ mod 3
0	5	2, 6	1, 2
1	1	2, 6	1, 2
2	3	2, 6	1, 2

Initially X is empty, i.e. $X \equiv 0 \pmod 3$, so that initially Alice chooses 5.

In any case, we see that Bob is prevented on every turn from winning. □

3. Points A, B, C, D lie on sides EF, FG, GH, HE, respectively, of a parallelogram $EFGH$. Also $AC \perp EF$, $BD \perp FG$, $ABCD$ is cyclic, and Q is the point on AC such that $FQ \parallel BC$. Prove that $EQ \parallel DC$.

Proof 1. We will define Q' on AC, show $FQ' \parallel BC$, so that $Q' = Q$. Then similarly show $EQ' \parallel DC$.



Let

$$\begin{aligned}K &= \text{circumcircle}(ABCD)\\ \{X\} &= GF \cap K \setminus \{B\}\\ \{Y\} &= EH \cap K \setminus \{D\}\\ Q' &= XY \cap AC.\end{aligned}$$

Then

$DBXY$ is cyclic (circumcircle is K) with adj. $\angle D = \angle B = 90°$

$\therefore$ $DBXY$ is a rectangle

$AFXQ'$ is cyclic (opp. $\angle A = \angle X (= 90°)$ are suppl.)

$AXBC$ is cyclic (circumcircle is K)

$\therefore$ $\angle FQ'A = \angle FXA$, angles on $\overset{\frown}{FA}$ in cyclic $AFXQ'$

$= \angle BCA$, int. opp. angle of ext. angle of cyclic $AXBC$

$\therefore$ $FQ' \parallel BC$, (corr. angles)

$\therefore$ $Q' = Q$, since Q' on AC

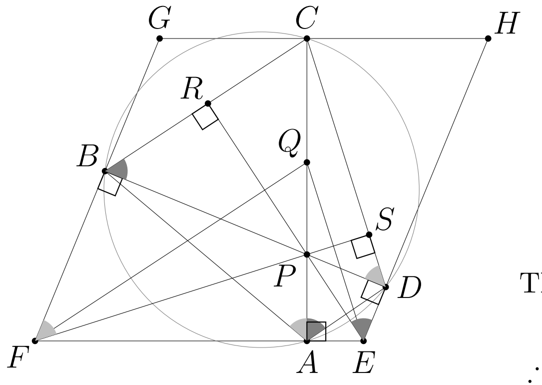
Similarly, (map: $F \to E, X \to Y, B \to D$)

$\angle EQ'A = \angle EYA$, angles on $\overset{\frown}{EA}$ in cyclic $AEYQ'$

$= \angle DCA$, int. opp. angle of ext. angle of cyclic $AYDC$

$\therefore$ $EQ = EQ' \parallel DC$, (corr. angles) □

Proof 2. This time we show $FQ \parallel BC$ implies $P = \text{orthocentre}(EFQ)$ implies $EQ \parallel DC$.



Let

$K = \text{circumcircle}(ABCD)$

$\ell = \text{line through } F, \parallel BC$

$Q = \ell \cap CA$

$P = BD \cap CA$

$R = EP \cap BC.$

$S = FP \cap DC.$

Then

$AEDP$ is cyclic (opp. $\angle A = \angle D (= 90^\circ)$ are suppl.)

$\therefore\ \angle DBR = \angle DBC$, same angle
$= \angle DAC$, angles on $\overset{\frown}{DC}$ in cyclic $ABCD$
$= \angle DAP$, same angle
$= \angle DEP$, angles on $\overset{\frown}{DP}$ in cyclic $AEDP$
$= \angle DER$, same angle

$\therefore\ BRDE$ is cyclic

$\therefore\ \angle BRE = \angle BDE$, angles on $\overset{\frown}{BE}$ in cyclic $BRDE$
$= 90^\circ$

Similarly, (map: $B \leftrightarrow D, E \to F, R \to S$)

$\angle BDS = \angle BFS$, $DSBE$ cyclic, $\angle DSF = \angle DBF = 90^\circ$

Now, $EP = ER \perp BC \parallel FQ$

$\therefore\ EP, AP$ are altitudes of $\triangle EFQ$

$\therefore\ P = \text{orthocentre}(EFQ)$

$\therefore\ FP$ is an altitude of $\triangle EFQ$

$\therefore\ FP \perp EQ$ and $FP \perp DC$

$\therefore\ EQ \parallel DC$ $\square$

4. Counters are placed, one at a time, in the unit squares of an $n \times n$ grid such that:

   (i) a counter can only be placed in an empty unit square,

   (ii) the first counter can be placed in any unit square,

   (iii) each subsequent counter can only be placed in a unit square S if

      the sum of the number of counters already in the same row as S and the number of counters already in the same column as S is odd.

   For each $n \geqslant 2$, find the smallest possible number of empty unit squares remaining after a sequence of such counter placements.

5. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be (not necessarily positive) real numbers, where $n \geqslant 2$.
   Let K, L be the maximum and minimum, respectively, of $b_1, b_2, \dots, b_n$.
   Prove that $\sum_{i<j} a_i a_j |b_i - b_j| \leqslant \frac{1}{2}(K - L)(a_1 + a_2 + \cdots + a_n)^2$,
   where $\sum_{i<j}$ denotes the summing over all pairs (i, j) such that $1 \leqslant i < j \leqslant n$.