1. Given any finite string made up of As and Bs, an occurrence of AB may be replaced with $BBAA$. Starting from any finite string of As and Bs, is it always possible to perform a number of such replacements so that all Bs appear to the left of all As?

Solution (Ivan Guo)

Answer: No, it's not always possible move all Bs to the left of all As.

We will prove that if a string contains at least one of AAB or ABB, then it will always contain at least one of AAB or ABB.

If a string has AAB, then either a move doesn't affect this, or it changes AAB to $ABBAA$ which contains ABB. Similarly, if a string has ABB, then either a move doesn't affect this, or it changes ABB to $BBAAB$ which contains AAB. This completes the proof.

2. A positive integer m is given to Alice and Bob. They play a game with the following rules.

To start the game, Alice writes a single nonzero digit on a board. Bob and Alice then take turns writing a single digit at either end of the current number on the board. A zero may be written at the end but not at the start of the number. Bob wins and the game ends if at any time the number on the board is divisible by m.

(i) What is the smallest value of m such that Alice can prevent Bob from ever winning?

(ii) Now suppose that Alice may start the game with any positive integer. All other rules remain the same. What is the smallest value of m such that Alice can prevent Bob from ever winning?

Solution (Brenton Gray, Mikaela Gray)

Answers: (i) $m = 12$, (ii) $m = 11$.

(i) If $m \leq 10$, then Bob has the winning strategy. After Alice writes the first digit, Bob has ten choices of digit (0 to 9) to write at the end of the number. The numbers so formed will be ten consecutive integers, one of which must thus be divisible by m and Bob chooses accordingly. If $m = 11$, whatever digit d Alice writes at the start, Bob on his first turn also writes d at the start (or end) to form the two-digit number dd which is divisible by 11.

If $m = 12$, then Alice now has the winning strategy. Say a number is a *12-blocker* if it is congruent to $5 \pmod 6$. Alice starts with 5 which is a 12-blocker.

We will show that if the number on the board, n say, at the end of Alice's turn is a 12-blocker, then Bob will not be able to form a number divisible by 12 on his next turn. For Bob's new number to be divisible by 12, it is necessary to add an even digit, e say, at the end. Now $n \equiv 5 \pmod 6$ implies $10n \equiv 50 \pmod{60}$, which implies $10n + e \equiv 2 + e \pmod{12}$. Since no single digit e is congruent to $10 \pmod{12}$, Bob cannot form a number divisible by 12.

It remains to show that Alice can always write a 12-blocker on her turn. By adding a 1, 3 or 5 to the end of Bob's number, Alice can yield all possible odd congruence classes modulo 6, so she can always form a number which is $5 \pmod 6$.

(ii) By the solution to (i) we know that the only candidates for m are 11 and 12.

We will show that Alice has a winning strategy for $m = 11$. Say a number is an *11-blocker* if it has an odd number of digits and is congruent to $-1 \pmod{11}$. Alice first writes 120 which is an 11-blocker. If Bob adds a digit e to the start of an an 11-blocker, n say, this gives $10^k e + n$ where k is odd. Since $10^k \equiv -1 \pmod{11}$, $10^k e + n \equiv -e - 1 \pmod{11}$. If Bob adds a digit e to the end of n, this gives $10n + e \equiv e + 1 \pmod{11}$. Neither $e + 1$ nor $-e - 1$ can be divisible by 11 for some digit e and thus Bob cannot form a number divisible by 11.

It remains to show that Alice can always write an 11-blocker on her turn. Starting with an 11-blocker n, if Bob writes the digit e, Alice copies and writes the same digit in the same position on her turn. This new number is an 11-blocker as the combination ee is always a multiple of 11, and also $100n = n \pmod{11}$.

Remark.

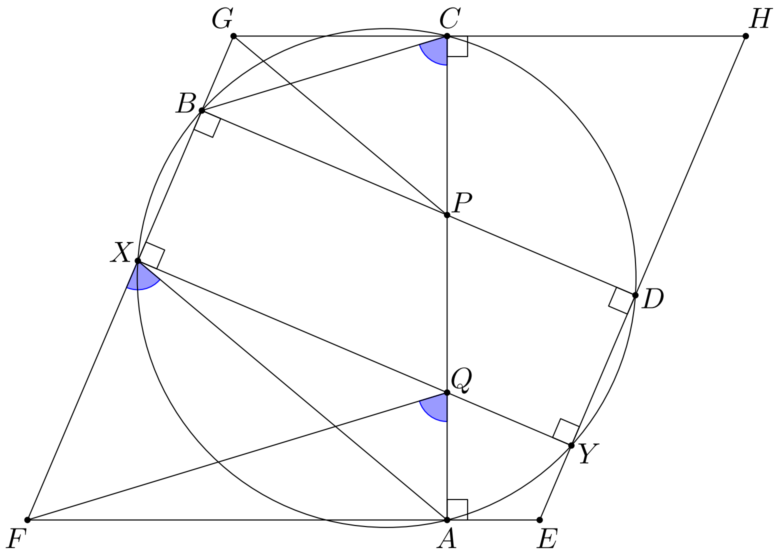
- Another way to present the strategy in (i) is as follows. Given the number x, Alice considers whether $100x + 10 + d$ is a multiple of 12 for some digit d. If a number is a multiple of 12, then its last two digits must be a multiple of 4. So there are three cases:
  - $100x + 12$ is a multiple of 12. Then $100x + 50 + d$ is not a multiple of 12 for any digit d. Alice can write the digit 5 at the end.
  - $100x + 16$ is a multiple of 12. Then $100x + 30 + d$ is not a multiple of 12 for any digit d. Alice can write the digit 3 at the end.
  - $100x + 10 + d$ is not a multiple of 12 for any digit d. So Alice can write the digit 1 at the end.
- An alternative strategy for (ii) is as follows. Suppose Bob's number B has an even number k of digits and is equal to say $c \neq 0 \pmod{11}$. If $c \neq 10 \pmod{11}$. Then Alice writes the digit $10 - c$ at start of B, so her number is $(10 - c)10^k + B$. Since k is even, this must be an 11-blocker. If $c = 10$, then Alice writes 9 at the end of B, making $10B + 9$ which is also an 11-blocker.

3. Points A, B, C, D lie on sides EF, FG, GH, HE, respectively, of a parallelogram $EFGH$. Suppose that $AC \perp EF$, $BD \perp FG$, and $ABCD$ is cyclic. Let Q be the point on AC such that $FQ \parallel BC$.

Prove that $EQ \parallel DC$.

Solution 1 (Angelo Di Pasquale)

Let circle $ABCD$ intersect FG for a second time at X and EH for a second time at Y. Let $Q' = XY \cap AC$. We shall prove that $Q = Q'$.



Since $DBXY$ is cyclic with right angles at B and D, it is a rectangle. Using cyclic quadrilaterals $AFXQ'$ and $AXBC$, we have

$$\angle FQ'A = \angle FXA = \angle BCA.$$

Hence $FQ' \parallel BC$, so $Q = Q'$. An analogous argument shows that

$$\angle EQA = \angle EYA = \angle DCP,$$

so $EQ \parallel DC$ as required.

Remark. There are variants of this solution which can be summarised as follows:
Let $X \in FG$ and $Y \in EH$ such that X, Q, Y are collinear. Then, using $BC \parallel FQ$, we obtain

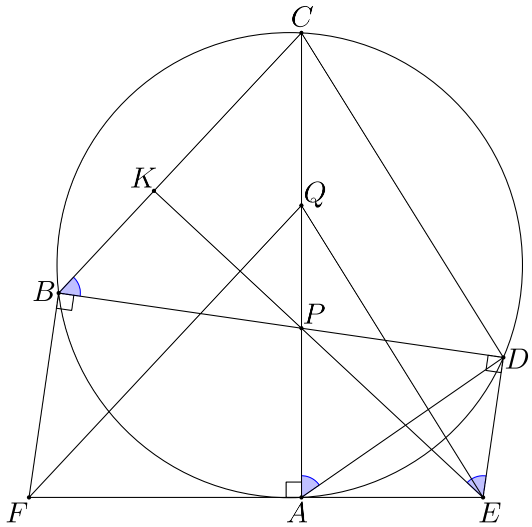
$$QX \perp FG \iff X \in (ABCD),$$

so assuming one would give the other. Finally, the problem can be completed by

$$\big\{QX \perp FG,\ X \in (ABCD)\big\} \iff \big\{QY \perp EH,\ Y \in (ABCD)\big\} \implies DC \parallel EQ.$$

Solution 2 (Angelo Di Pasquale)

Points G and H have been erased since they are not needed. We will drop the assumption that $FQ \parallel BC$ and prove that if $ABCD$ is cyclic, then $FQ \parallel BC$ if and only if $EQ \parallel DC$. Let $K = PE \cap BC$.



Since $ABCD$ and $AEDP$ are cyclic, we have

$$\angle DBC = \angle DAC = \angle DEP.$$

Hence $BKDE$ is cyclic, and so $\angle BKE = \angle BDE = 90^\circ$. Therefore

$$FQ \parallel BC \iff FQ \perp EP \iff P = \text{orthocentre}(EFQ)$$

where the last equivalence is due to $QA \perp EF$. Similarly $EQ \parallel DC$ if and only if $P = \text{orthocentre}(EFQ)$.

Hence $FQ \parallel BC$ if and only if $EQ \parallel DC$, as desired.

Remark. The statement $P = \text{orthocentre}(EFQ)$ in the argument can be replaced by other equivalent statements, for example:

- $Q = \text{orthocentre}(EFP)$;
- $AP \times AQ = AE \times AF$;
- $\triangle AFQ \sim \triangle APE$;
- $FQ'EP$ is cyclic, where Q' is the reflection of Q about A.

The problem can then be completed by similar angle chases.

4. Counters are placed, one at a time, in the unit squares of an $n \times n$ grid according to the following rules:

   - a counter can only be placed in an empty unit square
   - the first counter can be placed in any unit square
   - each subsequent counter can only be placed in a unit square S if the number of counters already in the same row as S and the number of counters already in the same column as S sum to an odd number.

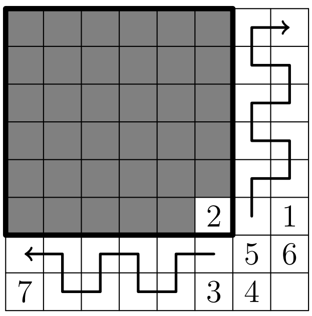
   For each value of $n \geq 2$, find the smallest possible number of empty unit squares remaining after a sequence of such placements.

Solution (Brenton Gray)

Let $f(n)$ be the minimum number of vacant squares remaining. We claim that $f(n) = 1$ if n is even, $f(n) = n - 1$ if n is odd.

We first show that $f(n) > 0$. Consider any board with a counter in every square except for one vacant square. The number of counters in each of the row and column related to the vacant square is $n - 1$ which sums to $2(n - 1)$, an even number, and so the final counter may not be placed (noting that the final counter is not the first counter as $n > 1$). Thus $f(n) \neq 0$.

Suppose $n = 2k$ is even. We shall show that $f(2k) = 1$ for all k by induction. It suffices to find a sequence of placements that fill all squares except the bottom right corner. The 2×2 case can be achieved by placing counters in the top left, top right and then bottom left squares. Assume that we have a suitable sequence of placements in a $2k \times 2k$ chessboard. Extend the $2k \times 2k$ chessboard with two extra rows and columns to form a $(2k+2) \times (2k+2)$ chessboard as shown in the figure below. First perform the sequence of placements as indicated by the two independent arrows, noting one arrow ends a square sooner than the other. The final numbering 1 to 7 shows how to complete the placements so that the bottom right square is the only vacant square remaining. Thus $f(2k+2) = 1$ which completes the induction for the even case.



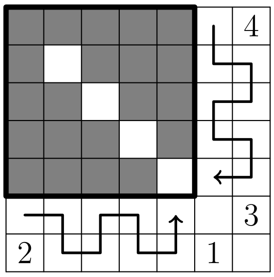
Now suppose $n = 2k + 1$ is odd. First we show that $f(2k + 1) \geq 2k$. We note that after the first counter is placed that the numbers of the counters in that row and column are both odd. All other rows and columns have an even number of counters (zero). For a subsequent counter to be placed in a square, the sum of counters in the corresponding row and column is odd. Or more specifically the number of counters in each of the row and column are of different parity. After placing a counter the number of counters in the corresponding row and column have their parity switched but nonetheless continue to be of different parity. Thus after the first and every subsequent placement the number of rows and columns with an odd number of counters remains unchanged and is precisely two.

Suppose there are strictly less than $2k$ vacant squares. Then there must be at least two full rows and two full columns. But then there are at least four rows and columns with and odd number of counters, which is a contradiction. Thus $f(2k+1) \geq 2k$.

We shall show that $f(2k+1) = 2k$ by induction on k. Let our induction hypothesis be that we can find a sequence of placements leaving $2k$ vacant squares on the main diagonal other than the top left square. The 3×3 case is given below.

5	1	2
4		3
7	6	

Now assume our hypothesis is true for some k and consider the $(2k+3) \times (2k+3)$ chessboard as shown below. Starting with the position from our induction hypothesis for the $(2k+1) \times (2k+1)$ chessboard first add counters as indicated by the two independent arrows. The final numbering 1 to 4 shows how to complete the placements so that in the new larger chessboard there are $2(k+1)$ vacant squares, each on the main diagonal other than the top left hand square, thus proving our induction hypothesis. Therefore $f(2k+1) = 2k$.



Remark.

- Alternative construction for the n even case: split the grid into 2×2 subgrids and fill each 2×2 subgrid in a zigzag-like manner from top to bottom. For example, the squares of a 6×6 grid would be filled in the order indicated below.

1	5	4	9	8	13
2	3	6	7	10	11
25	20	21	16	17	12
23	22	19	18	15	14
24	29	28	33	32	
26	27	30	31	34	35

- An alternative induction to give the even pattern from the preceding odd pattern: We fill in

$$(1+2l, n), (2+2l, n), (2+2l, 2+2l), (n, 2+2l), (n, 3+2l), (3+2l, 3+2l)$$

for $l = 0, \ldots, (n-4)/2$, then $(n-1, n)$ and $(n, 1)$.

5. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be (not necessarily positive) real numbers, where $n \geq 2$. Let K be the maximum of and L be the minimum of $b_1, b_2, \ldots, b_n$.

Prove that
$$\sum_{i<j} a_i a_j |b_i - b_j| \leq \tfrac{1}{2}(K-L)(a_1 + a_2 + \cdots + a_n)^2.$$

Note that $\displaystyle\sum_{i<j}$ refers to summing over all pairs (i, j) such that $1 \leq i < j \leq n$.

Solution 1 (William Steinberg)

Note that the indices $1, 2, \ldots, n$ can be permuted without changing the problem as long as the correspondence between a_i and b_i is maintained for each i. Let's reorder the two sequences simultaneously to maintain this correspondence, while sorting the b_i so that $b_1 \leq \cdots \leq b_n$. Then

$$\begin{aligned}2 \times (\text{RHS} - \text{LHS}) &= (b_n - b_1)\Big(\sum_{i=1}^{n} a_i\Big)^2 - 2\sum_{i<j} a_i a_j (b_j - b_i)\\ &= \Big(\sum_{k=1}^{n-1}(b_{k+1} - b_k)\Big)\Big(\sum_{i=1}^{n} a_i\Big)^2 - 2\sum_{i<j}\sum_{k=i}^{j-1} a_i a_j (b_{k+1} - b_k).\end{aligned}$$

Let's fix k and focus on terms that involve $(b_{k+1} - b_k)$. The first term will simply give a multiplier of $(\sum_{i=1}^{n} a_i)^2$. The second term is a sum over the triples (i, j, k) that satisfy $i \leq k < j$. So for a fixed k, the second terms gives a multiplier of $2\sum_{i,j} a_i a_j$ where $i \leq k$ and $j > k$. Hence

$$\begin{aligned}2 \times (\text{RHS} - \text{LHS}) &= \sum_{k=1}^{n-1}(b_{k+1} - b_k)\bigg(\Big(\sum_{i=1}^{n} a_i\Big)^2 - 2\sum_{i\leq k<j} a_i a_j\bigg)\\ &= \sum_{k=1}^{n-1}(b_{k+1} - b_k)\bigg(\sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j - 2\sum_{i\leq k<j} a_i a_j\bigg).\end{aligned}$$

Again fix k and consider the pairs (i, j) with $i < j$, if we take away those satisfying $i \leq k < j$, then we are left with either $k < i < j$ or $i < j \leq k$. Thus

$$\begin{aligned}2 \times (\text{RHS} - \text{LHS}) &= \sum_{k=1}^{n-1}(b_{k+1} - b_k)\bigg(\sum_{i=1}^{n} a_i^2 + 2\sum_{i<j\leq k} a_i a_j + 2\sum_{k<i<j} a_i a_j\bigg)\\ &= \sum_{k=1}^{n-1}(b_{k+1} - b_k)\bigg(\Big(\sum_{i=1}^{k} a_i\Big)^2 + \Big(\sum_{i=k+1}^{n} a_i\Big)^2\bigg)\\ &\geq 0,\end{aligned}$$

and we are done.

Remark. The equality cases can be characterised as follows: whenever $b_k \neq b_{k+1}$, then $\sum_{i\leq k} a_i = \sum_{i>k} a_i = 0$.

Solution 2 (Angelo Di Pasquale)

By symmetry we may assume without loss of generality that $b_1 \leq b_2 \leq \cdots \leq b_n$. Since the problem is invariant under translation of the b_i by a constant, we may assume that $b_1 = 0$.

Let $c_1 = b_1 = 0$ and $c_i = b_i - b_{i-1} \geq 0$ for $2 \leq i \leq n$. For $j > i$, we have $|b_i - b_j| = b_j - b_i = c_{i+1} + c_{i+2} + \cdots + c_j$. Thus the inequality becomes

$$\sum_{i<j} a_i a_j (c_{i+1} + c_{i+2} + \cdots + c_j) \leq \frac{c_1 + c_2 + \cdots + c_n}{2} \left(\sum_{i=1}^{n} a_i \right)^2. \tag{1}$$

We shall prove this for all $c_i \geq 0$ and all real a_i.

If $c_i = 0$ for all i, the problem is trivial. If $c_i > 0$ for some i, then since the inequality is invariant under scaling of the c_i by any positive real, we may assume that $c_1 + \cdots + c_n = 1$.

Let's hold all the a_i constant. Thus RHS(1) is constant. Let's try to maximise LHS(1).[1] If $c_i, c_j > 0$, then LHS(1) has the form

$$Ac_i + Bc_j + C = A(c_i + c_j) + C + (B - A)c_j$$

where A, B, C do not depend on c_i or c_j.

- If $B > A$, replacing (c_i, c_j) with $(0, c_i + c_j)$ increases LHS(1).
- If $B < A$, then replacing (c_i, c_j) with $(c_i + c_j, 0)$ increases LHS(1).
- If $B = A$, then replacing (c_i, c_j) with $(0, c_i + c_j)$ does not change LHS(1).

After at most $n - 1$ such moves, we arrive at the situation where $c_k = 1$ for some k and $c_i = 0$ for all $i \neq k$, and none of the moves ever decreased LHS(1). Putting this into (1), it suffices to show

$$\sum_{i<k\leq j} a_i a_j \leq \frac{1}{2} \left(\sum_{i=1}^{n} a_i \right)^2$$

$$\Leftrightarrow \quad \left(\sum_{i<k} a_i \right) \left(\sum_{i\geq k} a_i \right) \leq \frac{1}{2} \left(\sum_{i=1}^{n} a_i \right)^2.$$

But this follows from the inequality $XY \leq \frac{1}{2}(X + Y)^2 \iff X^2 + Y^2 \geq 0$ for all reals X and Y.

[1] One could also simply note that since LHS(1) is linear in the c_i, it is maximised at a vertex of the polytope defined by $c_i \geq 0$ and $c_1 + c_2 + \cdots + c_n = 1$. Thus $c_k = 1$ for some k and $c_i = 0$ for all $i \neq k$.

Statistics

Score Distribution/Problem

Mark/Problem	Q1	Q2	Q3	Q4	Q5
0	10	21	105	65	112
1	0	11	3	25	15
2	6	5	1	21	0
3	0	6	1	8	0
4	0	13	0	5	0
5	29	7	0	1	0
6	0	21	0	0	0
7	82	43	17	2	0
Average	**5.8**	**4.4**	**1.0**	**1.0**	**0.1**

The average score was 12.3.

Cuts for Gold, Silver and Bronze awards were 21, 15 and 13, repectively.