AUSTRALIAN MATHS TRUST | ASC. AMOC SENIOR CONTEST

DATE
Tuesday 20 August 2024

Time allowed: 4 hours
No calculators are to be used
Each question is worth 7 points

1. Consider a row of 1 000 001 coloured beads satisfying the following two conditions:

   - two beads have the same colour whenever there are exactly 99 beads between them
   - for every positive integer k less than or equal to 1 000 001, the kth bead from the left has the same colour as the kth bead from the right.

   Determine the maximum number of different colours of beads that could appear in the row.

2. For each positive integer n, the number $f(n)$ is also a positive integer. Furthermore:

   - $f(1) = 1$
   - $f(n+1) = n + 2 - f(f(n))$ for all positive integers n.

   Prove that $f(m) \geq f(n)$ for all positive integers $m \geq n$.

3. Let n be an integer greater than 5. Penny writes the numbers 2, 3, 4, ..., n on a blackboard. Then she erases all prime numbers greater than $n/3$. Penny wishes to rearrange the numbers remaining on the board in a circle so that each pair of neighbouring numbers has a common divisor greater than 1.

   Determine all values of n for which this is possible.

4. At a school, there are a number of clubs. A club is a set of students. Each club contains at least one student. A student may be in more than one club, but cannot be in every club.

   Surprisingly, for any two clubs A and B at the school, their union $A \cup B$ is also a club.

   Is it guaranteed that there is a club containing an even number of students?

   (Note: The union $A \cup B$ of two sets A and B is the set containing all elements that are in A or in B (or in both).)

5. Let ABC be a triangle with $AB < AC$ and let Γ be its circumcircle. The perpendicular line from A to BC meets Γ again at D. Let M be the midpoint of BC. Line DM meets Γ again at E. Suppose that P is a point on side BC with $PA = PC$. Suppose that line CE meets the circumcircle of triangle BPE again at F.

   Prove that the line through F parallel to BC, the perpendicular bisector of BC, and line AB are concurrent.

Optiver
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