1. Consider a row of 1 000 001 coloured beads satisfying the following two conditions:

   - two beads have the same colour whenever there are exactly 99 beads between them
   - for every positive integer k less than or equal to 1 000 001, the kth bead from the left has the same colour as the kth bead from the right.

   Determine the maximum number of different colours of beads that could appear in the row.

Solution

Answer: 51.

Since $1\,000\,001 \equiv 1 \pmod{100}$, we can label the beads

$$1,\ 2,\ 3,\ \ldots,\ 100,\ 1,\ 2,\ 3,\ \ldots,\ 100,\ \ldots\ ,\ 1,\ 2,\ 3,\ \ldots, 100,\ 1.$$

By the first condition, beads with the same label must have the same colour.

By the second condition, the colours must be symmetric with respect to the centre. So for each $k = 2, 3, \ldots, 50$, beads with label k must have the same colour as beads with label $102 - k$. Replacing the label $102 - k$ by k yields

$$1,\ 2,\ \ldots,\ 50,\ 51,\ 50,\ \ldots,\ 2,\ 1,\ \ldots\ ,\ 1,\ 2,\ \ldots,\ 50,\ 51,\ 50,\ \ldots,\ 2,\ 1.$$

More precisely, beads numbered $100n + r$ has label $\min(r, 100 - r)$.

Hence the answer is 51, which can be achieved by using different colours for different labels.

2. For each positive integer n, the number $f(n)$ is also a positive integer. Furthermore:

- $f(1) = 1$
- $f(n+1) = n + 2 - f(f(n))$ for all positive integers n.

Prove that $f(m) \geq f(n)$ for all positive integers $m \geq n$.

Solution

We will prove the following statement by strong induction. For all positive integers n,

$$f(n+1) - f(n) = 0 \text{ or } 1.$$

The base case holds since $f(1) = 1$ and $f(2) = 2$.

Suppose that for $i = 1, \ldots, k$, we have $f(i+1) - f(i) = 0$ or 1. Since $f(1) = 1$ and the function increases by 1 or 0 each time, we must have $f(k) \leq k$.

We now prove the statement for $k + 1$. By definition

$$\begin{aligned} f(k+2) - f(k+1) &= (k + 2 - f(f(k+1))) - (k + 1 - f(f(k))) \\ &= 1 - \big(f(f(k+1)) - f(f(k))\big). \end{aligned}$$

For the value of $f(f(k+1)) - f(f(k))$, there are two cases.

- If $f(k+1) = f(k)$, then $f(f(k+1)) = f(f(k))$.
- If $f(k+1) = f(k) + 1$, then by noting that $f(k) \leq k$, we may apply the induction hypothesis to $i = f(k)$ to obtain $f(f(k+1)) = f(f(k)+1) = f(f(k))$ or $f(f(k))+1$.

In both cases we have $f(f(k+1)) - f(f(k)) = 0$ or 1. Thus

$$f(k+2) - f(k+1) = 1 - (f(f(k+1)) - f(f(k))) = 0 \text{ or } 1,$$

completing the induction.

Therefore the function must be non-decreasing, as required.

3. Let n be an integer greater than 5. Penny writes the numbers 2, 3, 4, ..., n on a blackboard. Then she erases all prime numbers greater than $n/3$. Penny wishes to rearrange the numbers remaining on the board in a circle so that each pair of neighbouring numbers has a common divisor greater than 1.

   Determine all values of n for which this is possible.

Solution

Answer: it is possible for all n greater than 5 except for $n = 9, 10$ or 11.

For $n = 6, 7, 8$, we have $3 > \frac{n}{3}$ so all odd numbers are erased. The remaining numbers are all even and can be arranged arbitrarily.

Next we show that no arrangements exist for $n = 9, 10, 11$. Suppose an arrangement is possible and consider the placement of 3. The only other multiples of 3 are 6 and 9, so they must be the neighbours. But then the other neighbour of 9 cannot be a multiple of 3, a contradiction.

For $n \geq 12$, let's divide the numbers to be arranged into groups based on their largest prime divisor (so the groups are disjoint). For the group associated with a prime $p > 3$, note that $2p$ and $3p$ must be in the group since $p \leq \frac{n}{3}$. Arrange this group into the following block

$$B_p = [2p,\ \text{other multiples of } p,\ 3p]$$

where the multiples of p other than $2p$ and $3p$ are arranged in the middle arbitrarily. For the two groups containing 2 and 3, we combine them into a single block according to

$$B^* = [12,\ \text{other multiples of 3},\ 6,\ \text{other multiples of 2}].$$

Note that the adjacency rules are satisfied within each block.

Let $\hat{B}_p$ denote the reverse of B_p. A construction is then given by

$$B^*,\ B_5,\ \hat{B}_7,\ B_{11},\ \hat{B}_{13},\ \ldots$$

To check the validity of this construction, it suffices to note that every non-reversed block B_p starts with a multiple of 2 and ends with a multiple of 3, while B^* starts with a multiple of 6 and ends with a multiple of 2.

4. At a school, there are a number of clubs. A club is a set of students. Each club contains at least one student. A student may be in more than one club, but cannot be in every club.

Surprisingly, for any two clubs A and B at the school, their union $A \cup B$ is also a club.

Is it guaranteed that there is a club containing an even number of students?

(Note: The union $A \cup B$ of two sets A and B is the set containing all elements that are in A or in B (or in both).)

Solution 1

Answer: Yes.

For the sake of contradiction, assume every $|S_i|$ is odd. Let $S = \bigcup_i S_i$. From the union condition, S must be one of the sets, so $|S|$ must be odd.

Let $T_i = S \setminus S_i$, then $|T_i|$ must be even. Since the family of S_i is closed under unions, the family of T_i must be closed on intersections as $S \setminus (S_i \cup S_j) = (S \setminus S_i) \cap (S \setminus S_j) = T_i \cap T_j$.

The empty intersection condition on S_i implies that

$$\bigcup_i T_i = \bigcup_i (S \setminus S_i) = S \setminus \left(\bigcap_i S_i \right) = S.$$

Finally, the inclusion-exclusion principle gives

$$|S| = \sum_i |T_i| - \sum_{i<j} |T_i \cap T_j| + \sum_{i<j<k} |T_i \cap T_j \cap T_k| - \cdots + (-1)^{N+1} \left| \bigcap_i T_i \right|.$$

Recall that the family of T_i is closed under intersections, so the right-hand-side of the equation above only contains even numbers. But the left-hand-side is odd, contradiction.

Solution 2

Let $X = \{S_1, S_2, \ldots, S_N\}$. Suppose, for the sake of contradiction, that each $|S_i|$ is odd.

We prove by induction that the intersection of any k of the S_i contains an odd number of elements. This is true for $k = 1$. Assume it is true for some $k \geq 1$. If $A_1, \ldots, A_k, A_{k+1} \in X$, then using $|A \cup B| = |A| + |B| - |A \cap B|$ and the inductive assumption, we have

$$\begin{aligned}
|A_1 \cap \cdots \cap A_k \cap A_{k+1}| &= |A_1 \cap \cdots \cap A_k| + |A_{k+1}| - |(A_1 \cap \cdots \cap A_k) \cup A_{k+1}| \\
&\equiv 1 + 1 + |(A_1 \cap \cdots \cap A_k) \cup A_{k+1}| \pmod 2 \\
&\equiv |(A_1 \cap \cdots \cap A_k) \cup A_{k+1}| \pmod 2 \\
&= |(A_1 \cup A_{k+1}) \cap (A_2 \cup A_{k+1}) \cap \cdots \cap (A_k \cup A_{k+1})| \\
&= |B_1 \cap B_2 \cap \cdots \cap B_k| \\
&\equiv 1 \pmod 2
\end{aligned}$$

by the inductive assumption on $B_1, \ldots, B_k$ where $B_j = A_j \cup A_{k+1} \in X$ because X is closed under unions. This concludes the induction.

Hence $S_1 \cap S_2 \cap \cdots \cap S_N$ contains an odd number of elements and is therefore nonempty. Contradiction.

Remark. The induction here can be replaced by the dual form of the inclusion-exclusion principle:

$$\begin{aligned}\left|\bigcap_i S_i\right| &= \sum_i |S_i| - \sum_{i<j} |S_i \cup S_j| + \sum_{i<j<k} |S_i \cup S_j \cup S_k| - \cdots + (-1)^{N+1}\left|\bigcup_i S_i\right| \\ &\equiv N + \binom{N}{2} + \binom{N}{3} + \cdots + \binom{N}{N} \pmod 2 \\ &= 2^N - 1.\end{aligned}$$

Solution 3

Assume that such sets exist that are all of odd size. Renumber the sets so that $|S_1| \leq |S_2| \leq \ldots \leq |S_N|$. Note that $S_{N-1} \cup S_N = S_N$. Let $B = S_N \backslash S_{N-1}$. Since $|B| + |S_{N-1}| = |S_N|$, $|B|$ is even.

Consider any set S_i where $1 \leq i \leq N-2$ (if such exist). If S_i contains some, but not all elements in B, then $S_i \cup S_{N-1}$ would be larger than S_{N-1} but smaller than S_N, a contradiction. So S_i either contains no elements in B or all elements in B.

Call this action a *chop*: remove all elements in B from every set, and remove S_N. Since each set contains all or none of B and $|B|$ is even, each set still has odd size. Since the same elements are removed from every set, the sets remain closed over unions. Hence, there are now at most $N-1$ sets that fulfill the criteria given. Note that S_N which was removed had contained every element now remaining.

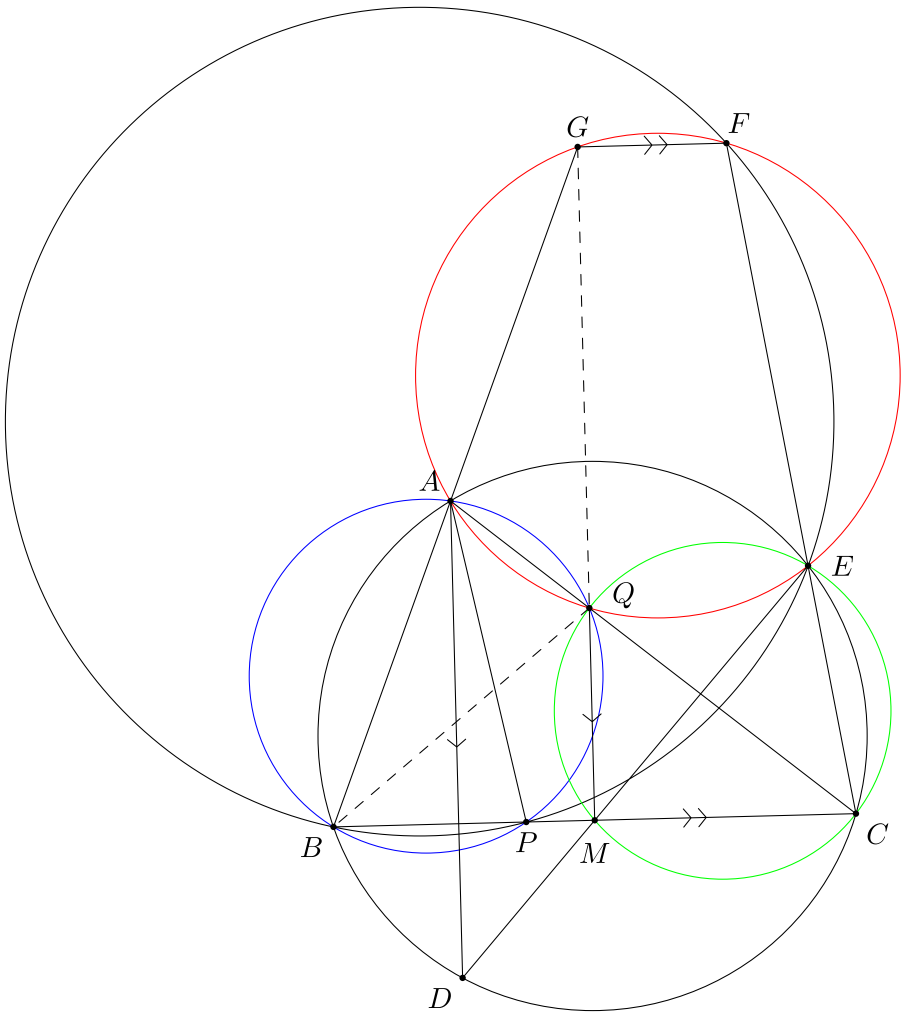
Repeating a similar *chop* with the new sets, again and again, eventually leads to one final set. This set has odd size, so is not empty. Now reversing the *chops* we find that the elements of this last remaining set had originally belonged to every set. This is a contradiction.

5. Let ABC be a triangle with $AB < AC$ and let Γ be its circumcircle. The perpendicular line from A to BC meets Γ again at D. Let M be the midpoint of BC. Line DM meets Γ again at E. Suppose that P is a point on side BC with $PA = PC$. Suppose that line CE meets the circumcircle of triangle BPE again at F.

   Prove that the line through F parallel to BC, the perpendicular bisector of BC, and line AB are concurrent.

Solution 1

We use directed angles modulo 180°. Let the line through F parallel to BC meet AB at G, and let the perpendicular bisector of BC meet AC at Q.



By angle chasing

$$\angle AQB = 2\angle ACB = \angle APB,$$

so $ABQP$ is cyclic (blue circle).

By the converse of the Radical Axis Theorem on circles $(ABQP)$ and $(BPEF)$ and the radical centre C, $AQEF$ is cyclic (red circle).

Since

$$\angle AGF = \angle ABC = \angle AEF,$$

$AEFG$ is cyclic. Therefore $AQEFG$ is a cyclic pentagon (red circle).

Next,

$$\angle CEM = \angle CAD = \angle CQM$$

implies that $CEQM$ is cyclic (green circle). Thus

$$\angle GQE = \angle GFE = \angle MCE = \angle MQE,$$

so G, Q, M are collinear, as required.

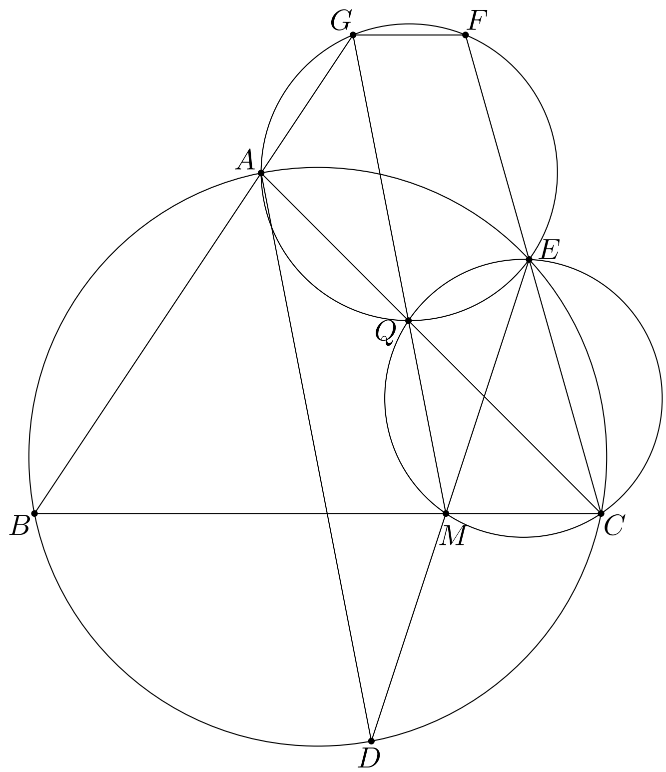
Remark. The solution repeatedly used the following lemma: Suppose $ABCD$ is cyclic. Let E be a point on line BC and F be a point such that $EF \parallel AB$. Then $EFDC$ is cyclic if and only if F lies on line AD.

Solution 2

Let $G = AB \cap MQ$ with Q as in the first solution.

Circles $ABPQ$ and $BPEF$ from the official solution serve to guarantee that $AQEF$ is cyclic. After this we can forget circles $ABPQ$ and $BPEF$. All that is needed from here is that $AD \parallel QM$ because this implies $CMQE$ is cyclic. E.g. $\angle ECA = \angle EDA = \angle EQM$.

From circles ABC and MQC, we see that E is the Miquel point of lines BG, BC, MG, AC. Hence $AQEG$ is cyclic. Combined with cyclic $AQEF$ yields that $AQEFG$ is cyclic. Finally Reim's theorem on circles $AQEFG$ and $CMQE$ yields $FG \parallel BC$, as desired. (Or angle chase: $\angle MGF = \angle QEC = 180° - \angle CMG$.) $\square$



Statistics

Score Distribution by Problem Number

Mark/Problem	Q1	Q2	Q3	Q4	Q5
0	0	44	18	63	113
1	0	18	23	24	2
2	4	6	9	5	2
3	7	2	5	3	0
4	8	6	2	5	1
5	4	4	12	2	0
6	9	6	21	4	2
7	95	41	37	21	7
Average	**6.3**	**3.2**	**4**	**1.9**	**0.6**

The average total score was 16.0 out of the maximum possible of 35.

Cuts for Gold, Silver and Bronze awards were 28, 22 and 15, respectively.[1]

[1] ASC awards are given approximately as follows:

- Gold: top 10%.
- Silver: top 25%.
- Bronze: top 50%
- Honourable Mentions are awarded to those who get full marks for at least one problem, but who miss out on a Gold, Silver or Bronze award.