INTERNATIONAL MATHEMATICS
# TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 42, Northern Spring 2021 (A Level)



**Note:** Each contestant is credited with the largest sum of points obtained for three problems.

**1.** The number $2021 = 43 \cdot 47$ is composite. Prove that after inserting any number of 8s between 20 and 21 the number remains composite. (4 marks)

**2.** In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile each child divides the current number of sweets in the pile by the number of children in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room.

Prove that the total number of sweets received by the boys does not depend on the order in which the children reach the pile. (5 marks)

**3.** There is an equilateral triangle $ABC$. Let $E$, $F$, and $K$ be points on side $AB$, side $AC$, and $AB$ produced, respectively, such that $AE = CF = BK$. Let $P$ be the midpoint of $EF$.

Prove that $\angle KPC$ is a right angle. (6 marks)

**4.** A tourist arrived on an island populated by 50 natives. The natives all stood in a circle and each of them announced the age of their left neighbour, followed by the age of their right neighbour. Each native is either a knight who told both numbers correctly or a knave who randomly increased one of the numbers by 1 and decreased the other by 1.

Is it always possible after hearing all the natives' announcements, for the tourist to determine which of the natives are knights and which are knaves? (7 marks)

**5.** In the centre of each cell of a chequerboard $M$ there is a pointlike light bulb. All the light bulbs are initially switched off. In a turn one chooses a straight line not passing through any light bulbs such that on one side of the line all light bulbs are switched off, and then all those light bulbs are switched on. At least one bulb must be switched on, in each turn. The task is to switch on all the light bulbs using the largest possible number of turns.

What is the maximum number of turns if:

(a) $M$ is a square of size $21 \times 21$? (4 marks)

(b) $M$ is a rectangle of size $20 \times 21$? (4 marks)

PLEASE TURN OVER

**6.** 100 tourists arrive at a hotel at night. They know that in the hotel there are single rooms numbered $1, 2, \ldots, n$, and among them $k$ are under repair and the other rooms are free to be used (but the tourists do not know which). One after another, the tourists check the rooms in an order of their choosing (which may differ for different tourists); each tourist takes the first room they find that is not under repair. The tourists don't know whether a room is already occupied until they check it. However it is forbidden to check an occupied room, and the tourists may coordinate their strategy beforehand to avoid this situation.

For each $k$ find the smallest $n$ for which the tourists may select their rooms for sure. (10 marks)

**7.** Let $p$ and $q$ be two coprime positive integers. A frog hops along an integer line such that on every hop it moves either $p$ units to the right or $q$ units to the left. Eventually, the frog returns to the initial point.

Prove that for every positive integer $d$ with $d < p + q$ there are two numbers visited by the frog which differ just by $d$. (12 marks)