



INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 42, Northern Spring 2021 (A Level)

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Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile each child divides the current number of sweets in the pile by the number of children in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room.

Prove that the total number of sweets received by the boys does not depend on the order in which the children reach the pile. (4 marks)

2. Does there exist a positive integer n such that for any real x and y there exist real numbers a_1, \dots, a_n satisfying

$$x = a_1 + \dots + a_n \quad \text{and} \quad y = \frac{1}{a_1} + \dots + \frac{1}{a_n} ? \quad (5 \text{ marks})$$

3. Let M be the midpoint of side BC of triangle ABC . The circle ω passes through A , touches BC at M , intersects side AB at the point D and side AC at the point E . Let X and Y be the midpoints of BE and CD respectively.

Prove that the circumcircle of triangle $MX Y$ touches ω . (5 marks)

4. There is a row of $100N$ ham sandwiches. A boy and his cat play a game. In one *action* the boy eats the first sandwich from any end of the row. In one *action* the cat either eats the ham from one sandwich or does nothing. The boy performs 100 actions in each of his turns, and the cat does only 1 action each turn. The boy starts first. The boy wins if the last sandwich he eats contains ham.

Is it true that the boy can win for any positive integer N no matter how the cat plays? (8 marks)

5. 100 tourists arrive at a hotel at night. They know that in the hotel there are single rooms numbered $1, 2, \dots, n$, and among them k are under repair and the other rooms are free to be used (but the tourists do not know which). One after another, the tourists check the rooms in an order of their choosing (which may differ for different tourists); each tourist takes the first room they find that is not under repair. The tourists don't know whether a room is already occupied until they check it. However it is forbidden to check an occupied room, and the tourists may coordinate their strategy beforehand to avoid this situation. For each k find the smallest n for which the tourists may select their rooms for sure.

PLEASE TURN OVER (8 marks)

6. Find at least one real number A such that for any positive integer n the distance between $\lceil A^n \rceil$ and the nearest square of an integer is 2.
(By $\lceil x \rceil$ we denote the smallest integer not less than x .) (10 marks)
7. An integer $n > 2$ is given. Peter wants to draw n arcs of length α of great circles on a unit sphere so that they do not intersect each other. Prove that the task
- (a) is possible, for all $\alpha < \pi + \frac{2\pi}{n}$. (6 marks)
- (b) is impossible, for all $\alpha > \pi + \frac{2\pi}{n}$. (7 marks)