INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 42, Northern Spring 2021 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. (a) A convex pentagon is partitioned into three triangles by nonintersecting diagonals. Is it possible for the centroids of these triangles to lie on a common straight line? (2 marks)

(b) The same question as (a), but for a non-convex pentagon. (2 marks)

2. (a) Maria has a balance scale that can indicate which of its pans is heavier or whether they have equal weight. She also has 4 weights that look the same but have masses of 1000, 1002, 1004 and 1005 g.

Can Maria determine the mass of each weight in 4 weighings? The weights for a new weighing may be chosen when the result of the previous ones is known. (2 marks)

(b) The same question as (a) except that the left pan of the scale is lighter by 1 g than the right one, so the scale indicates equality when the mass on the left pan is heavier by 1 g than the mass on the right pan. (2 marks)

3. For which n is it possible that a product of n consecutive positive integers is equal to a sum of n consecutive (not necessarily the same) positive integers? (5 marks)

4. It is well-known that a quadratic equation has no more than 2 roots.

Is it possible for the equation $\lfloor x^2 \rfloor + px + q = 0$ with $p \neq 0$ to have more than 100 roots? (By $\lfloor x^2 \rfloor$ we denote the largest integer not greater than x^2.) (5 marks)

5. Let O be the circumcentre of an acute triangle ABC, and M be the midpoint of AC. The straight line BO intersects altitudes AA_1 and CC_1 at points H_a and H_c, respectively. The circumcircles of triangles BH_aA and BH_cC have a second point of intersection K.

Prove that K lies on the straight line BM. (6 marks)