The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Tournament of the Towns Problems
Junior Paper: Years 8, 9, 10
Northern Autumn 2006 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. The area inside the circumcircle and outside the incircle of a regular 7-gon is equal to the area inside the circumcircle and outside the incircle of a regular 17-gon. Prove that sides of the polygons are equal. (3 points)

2. Ann draws a circle and represents each of her friends by a chord in this circle. If two of her friends know each other they are represented by chords which intersect inside or on the circle. If they do not know each other, they are represented by chords which do not intersect inside or on the circle. Ann believes that it is always possible to do so. Is she right? (5 points)

3. In the 3×3 square below, the sums of the three numbers in each row, in each column and on each diagonal are all equal.

a	b	c
d	e	f
g	h	i

   Prove that

   (a) $2(a + c + g + i) = b + d + f + h + 4e$; (3 points)

   (b) $2(a^3 + c^3 + g^3 + i^3) = b^3 + d^3 + f^3 + h^3 + 4e^3$. (3 points)

4. Three tangents to the incircle of an acute triangle are drawn, cutting off three right triangles and leaving behind a hexagon. Determine the sum of the diameters of the incircles of the three right triangles in terms of the inradius r of the original triangle and perimeter p of the hexagon. (6 points)

5. A rectangular piece of wrapping paper is of area 2 and red on only one side. It is used to wrap around a 1×1 square, without cutting, so that the square is completely wrapped on both sides in red. Clearly the shape of the piece of wrapping paper can be a 2×1 rectangle or a $\sqrt{2} \times \sqrt{2}$ square.

   (a) Find a third possible shape of the piece of wrapping paper. (4 points)

   (b) Prove that there exist infinitely many possible shapes of the piece of wrapping paper. (3 points)

6. For any positive integer n, let

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\frac{a_n}{b_n},$$

where $\dfrac{a_n}{b_n}$ is an irreducible fraction.
Prove that there exist infinitely many positive integers n such that $b_{n+1}<b_n$. (8 points)

7. The 52 playing cards of a standard deck are arranged face-up in a row. The order is visible to the audience which includes an assistant of the magician, but not to the magician himself. The assistant may name two cards and state how many other cards lie in between them. What is the minimum number of statements that the assistant must make so that the magician can deduce the relative order of the cards? (9 points)