The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

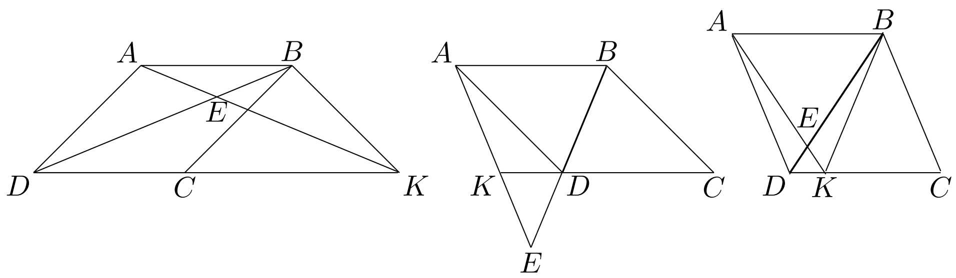
Tournament of the Towns Problems with Some Solutions and Some Hints

Junior Paper: Years 8, 9, 10
Northern Autumn 2007 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Let $ABCD$ be a rhombus. Let K be a point on the line CD, other than C or D, such that $AD = BK$. Let P be the point of intersection of BD with the perpendicular bisector of BC. Prove that A, K and P are collinear. (5 points)

Solution. The diagrams below show the possible locations of K relative to C and D, but in each case we have the same argument. Let the intersection point of AK with BD be E. Observe that $ABKD$ or $ABDK$ is a (convex) trapezium, and note that, since opposite angles of a trapezium are supplementary, a trapezium is a cyclic quadrilateral. Thus $\angle AKD = \angle ABD$ (angles at circumference standing on same arc AD). Also, $\angle ABD = \angle CBD$, since $\angle ABC$ of rhombus $ABCD$ is bisected by diagonal BD. So we have $\angle CBE = \angle CBD = \angle AKD = \angle EKC$ are equal angles standing on EC; thus $BCKE$ is also a cyclic quadrilateral. Hence, considering the arc BE, we have $\angle ECB = \angle EKB$, and $\angle EKB = \angle AKB = \angle ADB$ (angles at circumference standing on same arc AB). But $\angle ADB = \angle DBC = \angle EBC$ (alternating angles). Thus we have $\angle ECB = \angle EBC$, so that $\triangle ECB$ is isosceles and hence $EC = EB$. Thus E is equidistant from C and B, and so lies on the perpendicular bisector of BC. Hence, in fact, $E = P$, and so A, P and K are collinear as required.



Alternative method (using analytic geometry). Without loss of generality, let the sidelength of $ABCD$ be 1 unit, let D be the origin $(0, 0)$ and let the coordinates of A be (a, b). Since the sidelength of $ABCD$ is 1, we have $a^2 + b^2 = 1$. With these choices we have $B = (1 + a, b)$, $C = (1, 0)$ and $K = (1 + 2a, 0)$. Now the lines

$$AK: \quad y = -\frac{b}{1+a}\big(x - (1 + 2a)\big)$$
$$BD: \quad y = \frac{b}{1+a}x$$

intersect at E which is found to be

$$\Big(\frac{1+2a}{2}, \frac{b(1+2a)}{2(1+1)}\Big).$$

Now we find that

$$EC^2 - EB^2 = \frac{a(a^2+b^2-1)}{1+a} = 0, \quad \text{since } a^2+b^2 = 1.$$

Thus we have $EC^2 = EB^2$, and hence $EC = EB$, so that as above E lies on the perpendicular bisector of BC, and hence $E = P$, giving A, P and K collinear, as before.

2. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same numbers, prove that these three numbers are equal to one another. (3 points)

Solution. Let the numbers Peter thinks of be p_1, p_2 and p_3, the numbers Basil thinks of be b_1, b_2 and b_3, and the numbers both write down be w_1, w_2 and w_3. Note that each of $\gcd(w_1, w_2)$, $\gcd(w_2, w_3)$ and $\gcd(w_3, w_1)$ is equal to $\gcd(p_1, p_2, p_3)$. Similarly, each of $\operatorname{lcm}(w_1, w_2)$, $\operatorname{lcm}(w_2, w_3)$ and $\operatorname{lcm}(w_3, w_1)$ is equal to $\operatorname{lcm}(b_1, b_2, b_3)$. It follows that

$$w_1w_2 = \gcd(w_1, w_2)\operatorname{lcm}(w_1, w_2) = \gcd(w_2, w_3)\operatorname{lcm}(w_2, w_3) = w_2w_3,$$

so that $w_1 = w_3$, since the p_i and b_i are positive integers. Similarly, $w_1 = w_2$, and hence $w_1 = w_2 = w_3$.

Alternative solution. Let the numbers Peter thinks of be x, y and z. We assume to the contrary that $\gcd(x, y)$, $\gcd(x, z)$ and $\gcd(y, z)$ are not the same number. Then there must be a prime p such that the highest powers of p which divide these three numbers are not identical. Let the highest powers of p which divide x, y and z be a, b and c, respectively. Without loss of generality, assume that $a \le b \le c$. Then the highest powers of p which divide $\gcd(x, y)$, $\gcd(x, z)$ and $\gcd(y, z)$ will be a, a and b, respectively, and we have $a < b$. Now the highest powers of p which divides any of Basil's numbers must be b, and p^b will divide two of his least common multiples. It follows that the two sets of three numbers cannot be identical unless all three numbers in each set are the same.

(b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead? (3 points)

Solution. The answer is no, since we have the following counterexample. Peter may think of the numbers 1, 2, 2 and 2. Then he writes down the six greatest common divisors 1, 1, 1, 2, 2 and 2. Basil may think of the numbers 1, 1, 1 and 2. Then he writes down the six least common multiples 1, 1, 1, 2, 2 and 2. The two sets of numbers written down are identical, but the six numbers each of Peter and Basil think of are not all the same. Note that $6 = \binom{4}{2}$.

3. Michael is at the centre of a circle of radius 100 m. Each minute he will announce the direction in which he will be moving. Catherine can leave it as it is, or change it to the opposite direction. Then Michael moves exactly 1 m in the direction determined by Catherine. Does Michael have a strategy which guarantees that he can out of the circle, even though Catherine will try to stop him? (6 points)

Solution. Michael can escape. In the first move, he chooses any direction. Catherine cannot gain anything by reversing it. In each subsequent move, Michael chooses a direction which is perpendicular to the line joining his current position to the centre of circle. Again Catherine cannot gain anything by reversing it.
Let d_n be the distance of Michael from the centre of the circle after the n^{th} move. We have $d_1 = 1$ and $d_{n+1} = \sqrt{d_n^2 + 1}$.
We claim that $d_n = \sqrt{n}$ for all $n \geq 1$, and prove it by induction.
Our proposition is $P(n) : d_n = \sqrt{n}$.
$P(1)$ holds, since $d_1 = 1 = \sqrt{1}$.
Now assume $P(k)$ holds. Then $d_k = \sqrt{k}$ and

$$\begin{aligned} d_{k+1} &= \sqrt{d_k^2 + 1} \\ &= \sqrt{(\sqrt{k})^2 + 1} \\ &= \sqrt{k+1}. \end{aligned}$$

So $P(k+1)$ holds, if $P(k)$ holds.
Thus, by the Principle of Mathematical Induction, $P(n) : d_n = \sqrt{n}$ holds for all $n \geq 1$.
In particular, after 10 000 moves, Michael will arrive at the circumference of the circle, since $d(10\,000) = \sqrt{10^4} = 100$, so that on the 10 001 move, Michael will escape.

4. Two players take turns entering a symbol in an empty cell of a $1 \times n$ chessboard, where n is an integer greater than 1. Aaron always enters the symbol X and Betty always enters the symbol O. Two identical symbols may not occupy adjacent cells. A player without a move loses the game. If Aaron goes first, which player has a winning strategy? (7 points)

5. Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and replaced on the objects at random. We wish determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible? (8 points)

6. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.

   (a) Prove that if this is possible for some n, then it is possible for $2n$. (4 points)

   (b) Determine all n for which this is possible. (5 points)

7. For each letter in the English alphabet, William assigns an English word which contains that letter. His first document consists only of word assigned to the letter A. In each subsequent document, he replaces each letter of the preceding document by its assigned word. The fortieth document begins with "Till whatsoever star that guides my moving." Prove that this sentence reappears later in this document. (9 points)

Solution. Work backwards. The seven words given as starting the fortieth document are distinct, and hence correspond to distinct letters, i.e. the 39th document starts with seven distinct letters. Speculate on what the 38th document has to have been given what you have already learned, and then work from there.