The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Junior Paper: Years 8, 9, 10
Northern Autumn 2008 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. 100 queens are placed on a 100×100 chessboard. No queen is in position to take any other. Prove that there is at least one queen in each of the four 50×50 square quadrants. (4 points)

2. Each of four stones weighs an integer number of grams. Scales with two pans can show which of the pans has the heavier weight and the difference between the weights in the two pans, in grams. Can one determine the weights of all stones using 4 attempts with these scales, if at most one of the attempts may have error of 1 gram? (6 points)

3. Serge draws a triangle ABC and a median AD. Then he tells Iliya the lengths of median AD and side AC. Having this information, Iliya proves the following statement:

   $\angle CAB$ is obtuse, and $\angle DAB$ is acute.

   Find the ratio AD/AC (and prove Iliya's statement for any triangle with the same ratio). (6 points)

4. Baron Münchausen claims that he has a map of five towns of the country Oz. Every two towns are connected by a road, which doesn't lead to other towns. Each road intersects no more than one other road (and then no more than once). Roads are coloured yellow or red (according to the colour of bricks they are paved with), and when one walks around the perimeter of each town one observes that the colours of the emanating roads alternate. Can Baron Münchausen's story be true? (6 points)

5. We are given positive numbers $a_1, a_2, \dots, a_n$ such that $a_1 + a_2 + \cdots + a_n \le \frac{1}{2}$. Prove that $(1 + a_1)(1 + a_2) \cdots (1 + a_n) < 2$. (8 points)

6. Let ABC be a non-isosceles triangle. In the exterior of $\triangle ABC$, triangles $AB'C$ and $CA'B$ are constructed with equal base angles on the sides AC and BC, respectively. A perpendicular dropped to the segment $A'B'$ from the vertex C and a perpendicular bisector of the side AB meet at the point C_1. Find $\angle AC_1B$. (9 points)

7. Let $a_1, a_2, a_3, \dots$ be an infinite sequence, where $a_1 = 1$ and

$$a_n = \begin{cases} a_{n-1} + 1 & \text{if } \operatorname{god}(n) \equiv 1 \pmod 4 \\ a_{n-1} - 1 & \text{if } \operatorname{god}(n) \equiv 3 \pmod 4 \end{cases}$$

   where $\operatorname{god}(n)$ is the *greatest odd divisor* of n. Prove that

   (a) the number 1 appears infinitely many times in this sequence; (5 points)

   (b) every positive integer appears infinitely many times in this sequence. (5 points)

   (The first terms of this sequence are $1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, \dots$.)