The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Some Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2008 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. 100 queens are placed on a 100×100 chessboard. No queen is in position to take any other. Prove that there is at least one queen in each of the four 50×50 square quadrants. (4 points)

Solution. Firstly, there can be at most one queen in each rank (row) of the board. Hence, since there are 100 queens and 100 ranks, there is *exactly* one queen in each rank.
Similarly, there is *exactly* one queen in each file (column) of the board.
For a contradiction, assume one of the quadrants is empty. Without loss of generality, suppose the north-west quadrant is empty. Since the 50 queens in the 50 northern ranks are not in files of the north-west quadrant, they must be in eastern files. Hence, there are 50 queens in the north-east quadrant. Similarly, since the 50 queens in the 50 western files are not in ranks of the north-west quadrant, they must be in the southern ranks. Hence there are 50 queens in the south-west quadrant. However the squares in the SW and NE quadrants are covered by 99 positive-sloped diagonals. Since we have a total of 100 queens in these two quadrants, at least two of them lie on the same diagonal by the Pigeon Hole Principle, contradicting that no queen is in a position to take any other. Thus no quadrant can be empty, i.e. there is at least one queen in each of the four 50×50 square quadrants, as required.

Technically, the above is only half the story. One should really show that there is at least one way to place 100 queens as prescribed on the 100×100 chessboard. One way is for the set Q of queen positions to be as follows

$$Q = \{(i,j) \mid i \in \{1,2,\dots,100\}, j = 2i \bmod 101\}$$

where the $(1,1)$ square is in the SW corner and the $(100,100)$ square is in the NE corner. (The effect of the '$\bmod 101$' is: '*no effect*' for $1 \le i \le 50$, and '-101' for $51 \le i \le 100$.)

Observe that for these positions i ranges over all ranks $1,2,\dots,100$. So no two queens share a rank. Likewise, j ranges over all files $1,2,\dots,100$ (even-numbered files $2,4,\dots,100$ for $i \in \{1,2,\dots,50\}$ and odd-numbered files $1,3,\dots,99$ for $i \in \{51,52,\dots,100\}$). So no two queens share a file. Each positive-sloped diagonal has an equation of form

$$j = i + c,$$

where the constant c may be used to identify the diagonal, i.e. to show no two queens share a positively-sloped diagonal, it is sufficient to show that $j - i$ is different for each $(i,j) \in Q$. In fact,

$$
\begin{aligned}
j - i &= \begin{cases} 2i - i = i, & \text{if } i \in \{1,\dots,50\} \\ i - 101, & \text{if } i \in \{51,\dots,100\} \end{cases} \\
&\in \{1,\dots,50\} \cup \{-50,\dots,-1\},
\end{aligned}
$$

100 different positively-sloped diagonals. So, the queens are on different positively-sloped diagonals. Each negatively-sloped diagonal has an equation of form

$$j = -i + c,$$

where, again, the constant c may be used to identify the diagonal, i.e. to show no two queens share a negatively-sloped diagonal, it is sufficient to show that $j + i$ is different for each $(i, j) \in Q$. In fact,

$$
\begin{aligned}
j + i &= \begin{cases} 2i + i = 3i, & \text{if } i \in \{1,\dots,50\} \\ 3i - 101, & \text{if } i \in \{51,\dots,100\} \end{cases} \\
&\in \{3,6,\dots,150\} \cup \{52,55,\dots,199\},
\end{aligned}
$$

100 different negatively-sloped diagonals (since each member of $\{3, 6, \dots, 150\}$ is divisible by 3 and each member of $\{-52, -49, \dots, 199\}$ is congruent to -1 modulo 3). So, the queens are on different negatively-sloped diagonals.
Hence none of the queens with positions in Q are attacking one another, and so Q defines one possible configuration for the 100 queens.

2. Each of four stones weighs an integer number of grams. Scales with two pans can show which of the pans has the heavier weight and the difference between the weights in the two pans, in grams. Can one determine the weights of all stones using 4 attempts with these scales, if at most one of the attempts may have error of 1 gram? (6 points)

Solution. Let the four stones be A, B, C, D with masses a, b, c, d, respectively. As a **first attempt**, let us weigh B, C and D against A, A and B against C and D, A and C against B and D, and A and D against B and C, so that we have

$$-a + b + c + d = w \tag{1}$$
$$a + b - c - d = x \tag{2}$$
$$a - b + c - d = y \tag{3}$$
$$a - b - c + d = z \tag{4}$$

Then

$$
\begin{aligned}
(1) + (2) + (3) + (4): \quad & 2a = w + x + y + z \\
& \therefore a = \frac{w + x + y + z}{2} \\
(3) + (4): \quad & 2a - 2b = y + z \\
& \therefore b = a - \frac{y + z}{2} \\
(2) + (4): \quad & 2a - 2c = x + z \\
& \therefore c = a - \frac{x + z}{2} \\
(2) + (3): \quad & 2a - 2d = x + y \\
& \therefore d = a - \frac{x + y}{2}
\end{aligned}
$$

If there are no errors, this scheme works to determine the masses of the objects. If there is an error of 1 gram in one of the weighings, then we will be able to *detect* that an error has occurred since $w+x+y+z$ will not be divisible by 2. Additionally, if 2 divides $x+y$ and $x+z$ the error is in w, if 2 divides $x+y$ but not $x+z$ the error is in z, if 2 divides $x+z$ but not $x+y$ the error is in y, and if 2 divides $y+z$ but not $x+y$ the error is in x. However, we cannot tell from the above how to *correct* the error (should we add or subtract 1 to correctly adjust the weighing in error?).

As a **second attempt**, on the first weighing we could remove A from the pan, so that (1) becomes

$$b+c+d=w.$$

Then adding the four equations and isolating a gives

$$a=\frac{w+x+y+z}{3}.$$

Now, if $3 \mid w+x+y+z$ there is no error, and if $3 \nmid w+x+y+z$ we know there is an error and also the direction of the error (if $w+x+y+z \equiv 1 \pmod 3$, the error is positive, and if $w+x+y+z \equiv -1 \pmod 3$, the error is negative). The same scheme as above works to determine which of w,x,y,z is in error, and since we know the direction of the error, this time we can both *detect* and *correct* the error.

The question arises as to whether, in the second attempt, having a pan empty for one weighing is in the spirit of the question. If not, we may shift the weight B to the opposite pan for each weighing, which is equivalent to replacing b by $-b$ everywhere. So finally, as a **third attempt** we have: weigh C and D against B, A against B, C and D, A, B and C against D, and A, B and D against C, to obtain

$$-b+c+d=w \tag{5}$$
$$a-b-c-d=x \tag{6}$$
$$a+b+c-d=y \tag{7}$$
$$a+b-c+d=z \tag{8}$$

Then

$$(5)+(6)+(7)+(8): \qquad 3a=w+x+y+z \tag{9}$$
$$\therefore a=\frac{w+x+y+z}{3} \tag{10}$$
$$(7)+(8): \quad 2a+2b=y+z$$
$$\therefore b=\frac{y+z}{2}-a \tag{11}$$
$$(6)+(8): \quad 2a-2c=x+z$$
$$\therefore c=a-\frac{x+z}{2} \tag{12}$$
$$(6)+(7): \quad 2a-2d=x+y$$
$$\therefore d=a-\frac{x+y}{2} \tag{13}$$

If $3 \mid w+x+y+z$ then there is no error and a,b,c,d are determined as above. Otherwise, if

$$w+x+y+z \equiv \varepsilon \pmod 3 \text{ where } \varepsilon \in \{1,-1\}$$

then the *correction* is to add $-\varepsilon$ to one of w,x,y,z and then deduce a,b,c,d via (10)–(13).

Observe that if $2 \mid u+v$, i.e. $u+v \equiv 0 \pmod 2$ then

$$u \equiv v \pmod 2$$

Since (11)–(13) imply

$$y+z \equiv x+z \equiv x+y \equiv 0 \pmod 2$$

we have

$$x \equiv y \equiv z \pmod 2.$$

Also from (9) we have

$$\begin{aligned} &3a - w = x + (y+z) \\ &\therefore a - w \equiv x \pmod 2, \qquad\qquad \text{since } y+z \equiv 0 \pmod 2. \end{aligned}$$

Thus the one of $a-w, x, y, z$ that differs in parity from the other three is the one where the error lies ($a-w$ differing in parity indicates w is in error). This is a simpler *detection* scheme than the one given previously.

3. Serge draws a triangle ABC and a median AD. Then he tells Iliya the lengths of median AD and side AC. Having this information, Iliya proves the following statement:

   $\angle CAB$ is obtuse, and $\angle DAB$ is acute.

   Find the ratio AD/AC (and prove Iliya's statement for any triangle with the same ratio). (6 points)

Solution.

Background

Recall that a *locus* (plural: *loci*) is a set of points (generally of a geometrical shape) that satisfy a given property, e.g.

The *locus* of points that are a distance r from a point O is a circle centred at O and of radius r. Algebraically, if O has coordinates (x_0, y_0) then the circle is the locus of points (x, y) satisfying

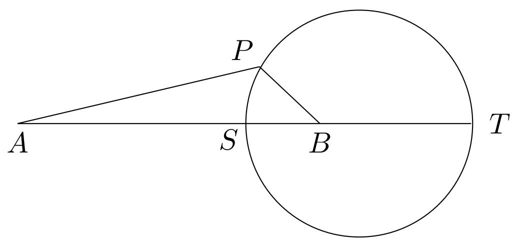
$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

which follows immediately by using Pythagoras Theorem to write the distance r in terms of the points (x, y) and (x_0, y_0).

As another example, the locus of points equidistant from two points P and Q is the perpendicular bisector of the two points.

If a line segment BC is the diameter of a circle K and on the semicircular arc $\overset{\frown}{BC}$ there lies a point A then $\angle BAC = 90°$.
Otherwise, if A lies *inside* K, $\angle BAC$ is *obtuse*, and if A lies *outside* K, $\angle BAC$ is *acute.*

In this problem, we also need the locus of points for which the ratio of distances to two given points is a constant. It turns out the locus is a circle. To see this, let the two points be $A(x_1, y_1)$ and $B(x_2, y_2)$, a point on the locus be $P(x, y)$ and the ratio of distances $d_{AP} : d_{BP} = \rho$, and suppose $\rho \neq 1$ (if $\rho = 1$ the locus is the perpendicular bisector of A and B). Then



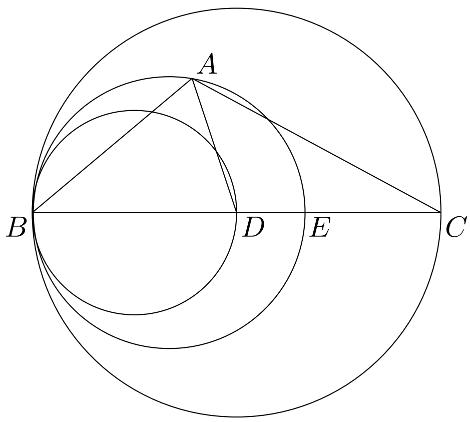
$$(x-x_1)^2 + (y-y_1)^2 = \rho^2\big((x-x_2)^2 + (y-y_2)^2\big).$$

Observe that after expansion and collection of terms on one side the coefficients of x^2 and y^2 are the same, so that after dividing by that common coefficient and doing completions of squares we will be able to put the equation in the form

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

for some constants x_0, y_0, R, which establishes that it's a circle. To find x_0, y_0, R, it's easier to go back to the geometry, rather than wade through the algebra. (We do enough algebra to convince ourselves it *is* a circle.) Observe that S and T are two points on a diameter of the circle, which one can show are given by $(A + \rho B)/(1 + \rho)$ and $(A - \rho B)/(1 - \rho)$, respectively; their midpoint is the centre $O(x_0, y_0)$ and then $R = OS$. This locus is known as a *Circle of Apollonius*.

Now, let's try the problem. For $\angle CAB$ to be obtuse, A must lie *inside* a circle with diameter CB; for $\angle DAB$ to be acute, A must lie *outside* a circle with diameter DB. Since, Iliya has only been given AD and AC and not the length of BC, he effectively only has the ratio AD/AC and no other positional information for the point A. Thus, Iliya only knows that A lies on a *Circle of Apollonius*, and since he claimed that he could prove that $\angle CAB$ was obtuse and $\angle DAB$ acute, the circle on which A lies is between the other two circles, i.e. we have the picture as shown. So we have that B lies on the *Circle of Apollonius* through A with distance ratio to points D and C of AD/AC, i.e.



$$\frac{AD}{AC} = \frac{BD}{BC} = \frac{1}{2}.$$

(Recall that AD is a median, so that D is the midpoint of BC, and hence $BD = \frac{1}{2}BC$.)

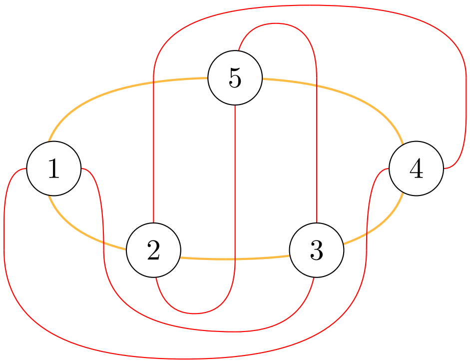
Finally, suppose $AD/AC = \frac{1}{2}$. Then A lies on a Circle of Apollonius with diameter FE where $F = B$ and E are external and internal points of DC with distance ratio $\frac{1}{2}$, i.e.

$$\begin{aligned}
F = \frac{-\frac{1}{2}C + D}{-\frac{1}{2} + 1} &= -C + 2D \\
&= D + D - C \\
&= D + \overrightarrow{CD} = B \\
E = \frac{\frac{1}{2}C + D}{\frac{1}{2} + 1} &= \tfrac{1}{3}C + \tfrac{2}{3}D, \quad \text{a point between } C \text{ and } D,
\end{aligned}$$

so that A lies on a circle of diameter BE sandwiched between circles of diameters BD and BC. Since ABC is a (non-degenerate) triangle, A does not lie on BC. And $\angle CAB$ is obtuse since A lies inside the circle with diameter BC, and $\angle BAD$ is acute since A lies outside the circle with diameter BD. This proves Iliya's statement for any triangle with the same ratio.

4. Baron Münchausen claims that he has a map of five towns of the country Oz. Every two towns are connected by a road, which doesn't lead to other towns. Each road intersects no more than one other road (and then no more than once). Roads are coloured yellow or red (according to the colour of bricks they are paved with), and when one walks around the perimeter of each town one observes that the colours of the emanating roads alternate. Can Baron Münchausen's story be true? (6 points)

Solution. Baron Münchausen's can be true: Number the towns $1, 2, 3, 4, 5$. Draw a path 1-2-3-4-5-1 representing the yellow road and thread through that path a path 1-3-5-2-4-1 representing the red road such that the 1-3 red road segment crosses the 1-2 yellow road segment, the 3-5 red road segment crosses the 4-5 yellow road segment, the 5-2 red road segment crosses the 2-3 yellow road segment, the 2-4 red road segment crosses the 5-1 yellow road segment, and the 4-1 red road segment crosses the 3-4 yellow road segment.



5. We are given positive numbers $a_1, a_2, \dots, a_n$ such that $a_1 + a_2 + \cdots + a_n \le \frac{1}{2}$. Prove that $(1 + a_1)(1 + a_2)\cdots(1 + a_n) < 2$. (8 points)

Solution.

Background

AM-GM Theorem. For a sequence of non-negative numbers $x_1, x_2, \dots, x_n$ its arithmetic mean (AM) is $\ge$ its geometric mean (GM), i.e.

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \dots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Binomial Theorem. $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$, where $\binom{n}{r} = \dfrac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1} \le n^r$.

We start by applying AM-GM to the sequence $1 + a_1, 1 + a_2, \dots, 1 + a_n$:

$$\begin{aligned}
\sqrt[n]{(1+a_1)(1+a_2)\cdots(1+a_n)} &\le \frac{1 + a_1 + 1 + a_2 + \cdots + 1 + a_n}{n} \\
&= 1 + \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad \text{since there are } n \text{ 1s} \\
&\le 1 + \frac{1}{2n}, \quad \text{since } a_1 + a_2 + \cdots + a_n \le \tfrac{1}{2} \\
\therefore (1+a_1)(1+a_2)\cdots(1+a_n) &\le \left(1 + \frac{1}{2n}\right)^n, \quad \text{since } f(x) = x^n \text{ is an increasing function for } x \ge 0 \text{ and } n \in \mathbb{N}, \text{ i.e. } 0 < u < v \implies u^n < v^n \\
&= 1 + n \cdot \frac{1}{2n} + \binom{n}{2} \frac{1}{(2n)^2} + \cdots + \binom{n}{r} \frac{1}{(2n)^r} + \cdots + \frac{1}{(2n)^n} \\
&= 1 + n \cdot \frac{1}{2n} + n^2 \cdot \frac{1}{(2n)^2} + \cdots + n^r \cdot \frac{1}{(2n)^r} + \cdots + n^n \cdot \frac{1}{(2n)^n} \\
&= 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} \\
&< \frac{1}{1 - \frac{1}{2}} = 2
\end{aligned}$$

6. Let ABC be a non-isosceles triangle. In the exterior of $\triangle ABC$, triangles $AB'C$ and $CA'B$ are constructed with equal base angles on the sides AC and BC, respectively. A perpendicular dropped to the segment $A'B'$ from the vertex C and a perpendicular bisector of the side AB meet at the point C_1. Find $\angle AC_1B$. (9 points)

7. Let $a_1, a_2, a_3, \dots$ be an infinite sequence, where $a_1 = 1$ and

$$a_n = \begin{cases} a_{n-1} + 1 & \text{if } \operatorname{god}(n) \equiv 1 \pmod 4 \\ a_{n-1} - 1 & \text{if } \operatorname{god}(n) \equiv 3 \pmod 4 \end{cases}$$

where $\operatorname{god}(n)$ is the *greatest odd divisor* of n. Prove that

(a) the number 1 appears infinitely many times in this sequence; (5 points)

(b) every positive integer appears infinitely many times in this sequence. (5 points)

(The first terms of this sequence are $1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, \dots$.)

Solution. Here's a start ...
Write $2^k \parallel n$ to mean $2^k \mid n$, but $2^{k+1} \nmid n$ (i.e. 2^k is the highest power of 2 that divides n), then we may decompose any natural number n as

$$n = 2^k t \text{ such that } 2^k \parallel n \text{ and } t = \operatorname{god}(n).$$

Let $\varepsilon_n = a_n - a_{n-1}$. Then observe that

$$\begin{aligned} \varepsilon_n &= \begin{cases} 1 & \text{if } \operatorname{god}(n) \equiv 1 \pmod 4 \\ -1 & \text{if } \operatorname{god}(n) \equiv 3 \equiv -1 \pmod 4 \end{cases} \\ &\equiv \operatorname{god}(n) \pmod 4 \end{aligned}$$

Now we tabulate ε_n according to its $n = 2^k t$ decomposition, entering ε_n (resp. a dot) in the k^{th} row if n is (resp. is not) of form $n = 2^k t$, where k and t are as above.

	$\varepsilon_n \in \{1, -1\}$														
$2^k t = n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t = 2^0 t$	1	.	-1	.	1	.	-1	.	1	.	-1	.	1	.	-1
$2t = 2^1 t$	.	1	.	.	.	-1	.	.	.	1	.	.	.	-1	.
$2^2 t$	.	.	.	1	.	.	.	.	.	.	.	-1	.	.	.
$2^3 t$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
a_n	1	2	1	2	3	2	1	2	3	4	3	2	3	2	1

Since n can be decomposed as $2^k t$, as prescribed above, in only one way, the value of ε_n appears in at most one row of the table, and since every n has such a decomposition, the value of ε_n appears in at least one row of the table. It follows that there is exactly one non-dot entry (i.e. either a 1 or a -1) in each column of the table.
By the definition of a_n,

$$a_n = a_{n-1} + \varepsilon_n,$$

for $n \geq 2$, and $a_1 = 1 = \varepsilon_1$. Thus $a_n = \sum_{j=1}^{n} \varepsilon_j$, i.e. a_n is the sum of all the 1 and -1 entries in the table up to the n^{th} column.
Observe that the sequence of values of t in every row is $1, 3, 5, 7, \dots$ which are alternately congruent to 1 and -1 *modulo* 4. So, the 1s and -1s alternate in every row.

(a) We are done if we can show $a_n = 1$ for $n = 2^m - 1, m \in \mathbb{N}$.

(b) Prove that a_n gets arbitrarily large. Then using (a), since a_n always returns to 1 after getting to a new high number, it must return through each finite number an infinite number of times. To do this properly, use a contradiction argument.