The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Junior Paper: Years 8, 9, 10
Northern Autumn 2009 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. There are 10 jugs of the same size with some milk in each. They are not necessarily equally filled. However, no jug is filled more than 10% of its capacity. It is possible, in one step, to choose a jug and pour out some milk equally into all other jugs.

   Prove that all jugs can be equally filled with milk after not more than 10 such steps. (4 points)

2. Misha has 1000 cubes of the same size. Each cube has a pair of opposite faces coloured white, another pair of opposite faces coloured blue and the third pair of faces coloured red. He puts all cubes together to get a bigger cube of size $10 \times 10 \times 10$ such that all smaller cubes are attached one to another by faces of the same colour.

   Prove that the bigger cube has a face of just a single colour. (6 points)

3. Find all $a, b \in \mathbb{N}$ such that $(a + b^2)(b + a^2)$ is an integer power of 2. (6 points)

4. Let $ABCD$ be a rhombus. Points P and Q lie on the sides BC and CD, respectively, with $BP = CQ$.

   Prove that the centroid of $\triangle APQ$ lies on BD. (6 points)

5. There are N weights with masses $1\,\text{g}, 2\,\text{g}, \dots, N\,\text{g}$, respectively. It is required to choose several weights (more than one) such that their total mass is equal to the average value of the masses of other weights.

   Prove that

   (a) it is possible to do, if $N + 1$ is a perfect square; (2 points)

   (b) if it is possible to do, then $N + 1$ is a perfect square. (7 points)

6. 2009 identical squares are placed on a grid plane, their sides always being along grid lines (the squares may overlap) and each covering exactly N grid squares (N being a perfect square). Then all grid squares which are covered by an odd number of squares are marked.

   Prove that the number of these grid squares marked is not less than N. (10 points)

7. Olya and Maksim have booked a travel tour to an archipelago of 2009 islands. Some of the islands are connected by 2-way routes using boats. While travelling they play a game. At first Olya can choose an island to fly to in order to start their tour. Then they travel by boats, taking it in turns to choose the next island, which must be one they have not been to before (Maksim takes first choice). When one cannot choose an island, they lose.

   Prove that Olya can win no matter how Maksim chooses. (14 points)