The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Some Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2009 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. There are 10 jugs of the same size with some milk in each. They are not necessarily equally filled. However, no jug is filled more than 10% of its capacity. It is possible, in one step, to choose a jug and pour out some milk equally into all other jugs.

   Prove that all jugs can be equally filled with milk after not more than 10 such steps. (4 points)

Solution. At each step, order the jugs by quantity from smallest volume of milk to largest volume. If at any step, all the jugs have the same quantity we are done. Otherwise, at Step i, let the quantities held by the jugs are

$$q_1^{(i)} < q_2^{(i)} < \cdots$$

where (i) indicates we are at Step i. At Step i, take one of the jugs with the second-smallest quantity

$$q_2^{(i)}$$

and distribute

$$\frac{q_2^{(i)} - q_1^{(i)}}{10}$$

to all the other jugs.

We claim at Step i (prior to the pouring), there are at least i jugs with quantity $q_2^{(i)}$. Since at Step 1. there is at least one jug with the least quantity the claim is true for $i = 1$.

So suppose that at Step i there are at least i jugs with quantity $q_2^{(i)}$. Then, after the pouring, since 9 jugs have had the their volumes of milk increased by the same quantity, their ordering with respect to volume is unchanged. However the jugs with the smallest quantity, now have

$$\begin{aligned} q_1^{(i+1)} = q_1^{(i)} + \frac{q_2^{(i)} - q_1^{(i)}}{10} &= \tfrac{9}{10}q_1^{(i)} + \tfrac{1}{10}q_2^{(i)} \\ &= q_2^{(i)} - \tfrac{9}{10}(q_2^{(i)} - q_1^{(i)}), \end{aligned}$$

where we recognise the last expression as the quantity now in the jug being poured from, i.e. the i previous jugs with the smallest quantity and the jug being poured from, all have the smallest quantity at Step $i+1$. So at least $i+1$ jugs have the smallest quantity $q_1^{(i+1)}$ at Step $i+1$.

It follows then, by induction, that at Step 10, if we haven't stopped sooner, at least 10 jugs, i.e. all 10, will have the same quantity (= the "smallest" quantity), which is after at most 9 operations.

2. Misha has 1000 cubes of the same size. Each cube has a pair of opposite faces coloured white, another pair of opposite faces coloured blue and the third pair of faces coloured red. He puts all cubes together to get a bigger cube of size $10 \times 10 \times 10$ such that all smaller cubes are attached one to another by faces of the same colour.

   Prove that the bigger cube has a face of just a single colour. (6 points)

Solution. Take one vertex of the large cube, and assign x-, y- and z- axes, in the standard way to the direction of the rays meeting at that vertex. For brevity, call a column of cubes whose axis is parallel to the x-axis, an x-column, and similarly define y- and z- columns. Also, for brevity, let us refer to the colours of faces of cubes along an axis parallel to the x-axis their x-colours, and similarly define y- and z- colours.

Firstly, consider a single column of 10 cubes (any orientation). Start at one end of the column, and let the second cube of the column be attached to the first on a face with colour c. Then the face opposite the second cube's c-coloured face is also c, and this is the face to which the third cube of the column is attached to the second cube of the column. Continuing in this way we see that all the faces of the cubes along the common axis of the column are of colour c. So it makes sense to talk of an x-colour for an x-column, since it is the same colour for each cube of the x-column. Similarly, we may define y- and z- colours of y- and z- columns, respectively.

Now we are ready to prove the claim.

If all cubes are in the same colour orientation, there is nothing to prove. Otherwise there are two adjacent cubes c_1 and c_2 with a different colour orientation. Suppose, without loss of generality, that they are in a common x-column C and that the common colour of their x-column faces is blue. Then the x-colour of C is blue.

Now, suppose x-column C is adjacent to x-column C', and without loss of generality, suppose that the common plane in which C and C' lie is parallel to the y-axis. Then, let the cubes c_1 and c_2 are attached to in C' be c_1' and c_2', respectively. Then the y-colours of c_1 and c_1', and of c_2 and c_2' are different from blue; and are different from each other, since c_1 and c_2 have different colour orientation, i.e. one of these y-colours is red and the other white. But then the x-colour of c_1' and c_2' is neither red nor white, and hence is blue. It follows that the x-colour of C' is blue.

Similarly, taking a column C'' (containing cubes c_1'' and c_2'' respectively adjacent to c_1 and c_1'), that is adjacent vertically to C, we have z-colours of c_1'' and c_2'' are white and red in some order (but in the opposite order to the y-colours of c_1' and c_2'). By the same reasoning as before, the x-colour of C''.

Continuing in this fashion from adjacent x-column to adjacent x-column we have that all x-columns have a common x-colour, and so the x-colours of the large cube is also common. Thus in all cases the large cube has at least two opposite faces (and hence at least one face) of a single colour.

3. Find all $a, b \in \mathbb{N}$ such that $(a+b^2)(b+a^2)$ is an integer power of 2. (6 points)

Solution. Consider two cases.

Case (i): $a = b$. Then

$$\begin{aligned}(a+b^2)(b+a^2) &= (a+a^2)^2 \\ &= a^2(a+1)^2 = 2^k \text{ for some } k \in \mathbb{N}\end{aligned}$$

$$\therefore a, a+1 \text{ are each integer powers of } 2$$

$$\implies \begin{cases}\big(a = 2^0 = 1 \text{ or } a \equiv 0 \pmod 2\big) \\ \quad \text{and} \\ a+1 \equiv 0 \pmod 2\end{cases}$$

$$\implies a = 1.$$

Checking, we find $a = b = 1$ gives

$$(a+b^2)(b+a^2) = (1+1^2)^2 = 2^2.$$

Case (ii): $a \neq b$. Then, without loss of generality, we may assume $a > b$, so that we have

$$\begin{aligned} a > b \geq 1 &> 0 \\ \text{and } a-1 > b-1 &\geq 0 \\ \therefore a(a-1) &> b(b-1) \\ \therefore a^2 - a &> b^2 - b \\ \therefore a^2 + b &> a + b^2. \end{aligned}$$

Now suppose $a + b^2 = 2^\ell$ and $a^2 + b = 2^m$, $\ell, m \in \mathbb{N}$. Then $\ell < m$ since $a + b^2 < a^2 + b$ and

$$\begin{aligned} 2^\ell(2^{m-\ell} - 1) &= 2^m - 2^\ell \\ &= a^2 + b - (a + b^2) \\ &= a^2 - b^2 - (a - b) \\ &= (a-b)(a+b-1). \end{aligned}$$

Now,

$$\begin{aligned} a - b &= a + b - 2b \\ &\equiv a + b \pmod 2. \end{aligned}$$

So $a-b$ and $a+b$ have the same parity, and hence $a-b$ and $a+b-1$ have opposite parity. Now the decomposition of an even number as the product of an even and an odd natural number is unique (any other decomposition as the product of natural numbers is necessarily the product of two even numbers).

Thus, either 2^ℓ equals $a-b$ or $a+b-1$. Now,

$$a + b^2 = 2^\ell = a - b$$

implies $b^2 = -b < 0$, which is impossible. Likewise,

$$a + b^2 = 2^\ell = a + b - 1$$

implies $b^2 = b - 1$, which is also impossible since $b > 1 \implies b^2 > b > b - 1$.

Thus there are no solutions if $a \neq b$.

Thus there is just one solution, namely $a = b = 1$.

4. Let $ABCD$ be a rhombus. Points P and Q lie on the sides BC and CD, respectively, with $BP = CQ$.

   Prove that the centroid of $\triangle APQ$ lies on BD. (6 points)

5. There are N weights with masses $1\,\text{g}, 2\,\text{g}, \ldots, N\,\text{g}$, respectively. It is required to choose several weights (more than one) such that their total mass is equal to the average value of the masses of other weights.

   Prove that

   (a) it is possible to do, if $N+1$ is a perfect square; (2 points)

   (b) if it is possible to do, then $N+1$ is a perfect square. (7 points)

Solution.

(a) Observe that

$$\begin{aligned}\sum_{m=1}^{n} &= \frac{n(n+1)}{2}\\ &= \frac{n^2+n}{2}\\ &= \frac{(n^2-1)+(n+1)}{2}\\ &= \mathrm{AM}(n+1, n+2, \ldots, n^2-1).\end{aligned}$$

So, if $N+1$ is a square, n^2 say, then one can take the masses $1\,\text{g}, 2\,\text{g}, \ldots, n\,\text{g}$ and their total mass is the varage of the masses $(n+1)\,\text{g}, \ldots, N = (n^2-1)\text{g}$.

(b) The total mass of the N objects is

$$\sum_{m=1}^{N} = \frac{N(N+1)}{2}.$$

Let the number of chosen objects be k and let

$$\begin{aligned}M &= \text{ mass of the } k \text{ chosen objects}\\ &= \text{ average mass of the remaining } N-k \text{ objects.}\end{aligned}$$

Then

$$\begin{aligned}\frac{N(N+1)}{2} &= M + (N-k)M\\ &= (N-k+1)M\\ \therefore 2M(N-k+1) &= N(N+1)\\ &> N(N+1) - k(k-1), \quad \text{since } k > 1 \text{ (given)}\\ &= N^2 + N - k^2 + k\\ &= N^2 - k^2 + N + k\\ &= (N+k)(N-k+1)\\ \therefore 2M &> N+k. \qquad (1)\end{aligned}$$

Now M is maximised for a given N when the $N-k$ not chosen objects are consecutive and largest (so that the chosen k weights are the k weights of least mass), i.e.

$$\begin{aligned} M &\le \frac{k+1+N}{2} \\ \therefore 2M &\le N+k+1. \end{aligned} \tag{2}$$

Now (1) and (2) imply

$$2M = N+k+1.$$

Further, equality in (2) is achieved exactly when the k lightest weights are chosen, i.e.

$$\begin{aligned} \sum_{m=1}^{k} m = \frac{k(k+1)}{2} &= m = \frac{N+k+1}{2} \\ \therefore \frac{k^2+k}{2} &= \frac{N+k+1}{2} \\ \therefore k^2 &= N+1. \end{aligned}$$

Thus, if it is possible then $N+1$ is a perfect square.

6. 2009 identical squares are placed on a grid plane, their sides always being along grid lines (the squares may overlap) and each covering exactly N grid squares (N being a perfect square). Then all grid squares which are covered by an odd number of squares are marked.

   Prove that the number of these grid squares marked is not less than N. (10 points)

7. Olya and Maksim have booked a travel tour to an archipelago of 2009 islands. Some of the islands are connected by 2-way routes using boats. While travelling they play a game. At first Olya can choose an island to fly to in order to start their tour. Then they travel by boats, taking it in turns to choose the next island, which must be one they have not been to before (Maksim takes first choice). When one cannot choose an island, they lose.

   Prove that Olya can win no matter how Maksim chooses. (14 points)