The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Some Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2010 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. A round coin may be used to construct a circle passing through one or two given points on the plane.

   Given a line on the plane, show how to use this coin to construct two points such that they define a line perpendicular to the given line.

   Note that the coin may not be used to construct a circle tangent to the given line.

   (4 points)

2. Petya has an instrument which can locate the midpoint of a line segment, and also the point which divides the line segment in the ratio $n : (n+1)$, for any $n \in \mathbb{N}$. Petya claims that with this instrument, he can locate the point which divides a line segment into two segments whose lengths are in any given rational ratio.

   Is Petya right? (5 points)

3. At a circular track, 10 cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at a point at the same time again, we say that they meet. No three of the cyclists meet at the same time.

   Prove that by the time every pair of cyclists have met at least once, each cyclist has had at least 25 meetings. (8 points)

4. A rectangle is divided into 2×1 and 1×2 dominoes. In each domino, a diagonal is drawn, such that no two diagonals have a common endpoint.

   Prove that exactly two corners of the rectangle are endpoints of these diagonals.

   (8 points)

5. For each side of a given pentagon, divide its length by the total length of all other sides.

   Prove that the sum of all the fractions obtained is less than 2. (8 points)

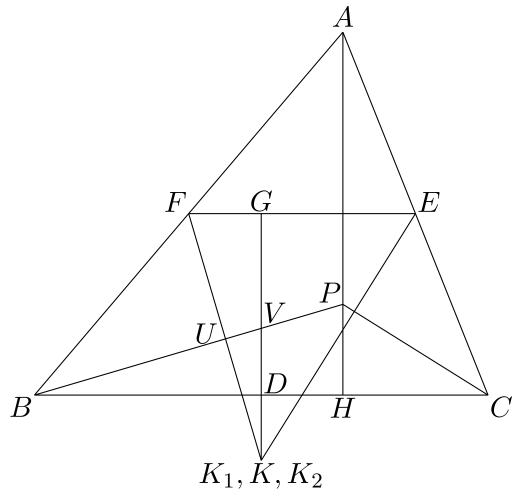
6. In acute $\triangle ABC$, an arbitrary point P is chosen on altitude AH, where the foot of the altitude is at H. Points E and F are the midpoints of sides CA and AB, respectively. The perpendiculars from E to CP and from F to BP meet at point K.

   Prove that $KB = KC$. (8 points)

   Solution. Let G be the point of intersection of FE with the perpendicular bisector of BC, i.e. GD is the perpendicular bisector of BC.

Let GD meet FK at K_1, and let FK and GD meet BP at points U and V, respectively. Since $\angle VUK_1 = \angle VDB = 90°$,

$$\begin{aligned}
\angle FK_1G &= \angle UK_1V, && \text{(same angle)}\\
&= 90° - \angle UVK_1\\
&= 90° - \angle BVD\\
&= \angle VBD\\
&= \angle PBH\\
\angle FGK_1 &= \angle PHB\\
\therefore \triangle FGK_1 &\sim \triangle PHB, && \text{by the AA Rule}\\
\therefore \frac{GK_1}{FG} &= \frac{HB}{PH}\\
\therefore GK_1 &= \frac{FG \cdot HB}{PH}.
\end{aligned}$$



If we analogously let the intersection of GD and EK be K_2, then similarly we have

$$GK_2 = \frac{EG \cdot HC}{PH}.$$

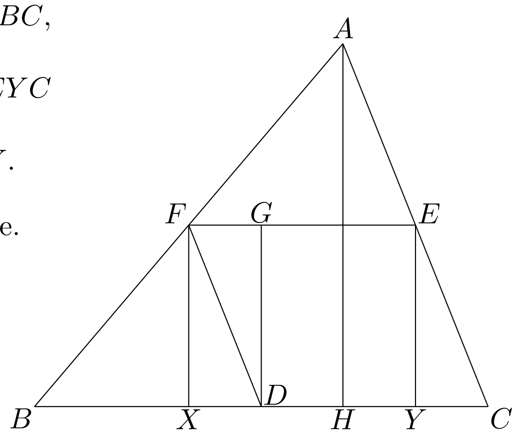
At this point we see that if we can show that $FG \cdot HB = EG \cdot HC$, it will follow that $K_1 = K_2 = K$, and this will lead to the required result. Thus we prove the following lemma.

Lemma. *Let A, B, C, E, F, G be as defined above. Then*

$$FG \cdot HB = EG \cdot HC.$$

Proof. Drop perpendiculars from F and E to BC to meet BC at X and Y, respectively. Then $FX \parallel AH$ with F the midpoint of BA, and hence X is the midpoint of BH, since transversals BA and BH are cut in equal ratios by parallels FX and AH.
Similarly, since $EY \parallel AH$ with E the midpoint of CA, Y is the midpoint of CH.
Since F and D are the midpoints of BA and BC, respectively, $FD \parallel AC$.
$\therefore$ the corresponding angles of $\triangle FXD$ and $\triangle EYC$ are equal.
Also, $FEYX$ is a rectangle, so that $FX = EY$.

$$\begin{aligned}
\therefore \triangle FXD &\cong \triangle EYC, && \text{by the AAS Rule.}\\
\therefore XD &= YC\\
&= YH\\
\therefore XH &= XD + DH\\
&= YH + HD\\
&= YD\\
\therefore FG \cdot HB &= XD \cdot 2XH\\
&= XH \cdot 2XD\\
&= YD \cdot 2YH\\
&= EG \cdot HC
\end{aligned}$$



□

So using the lemma we have

$$\begin{aligned} GK_1 &= \frac{FG \cdot HB}{PH} \\ &= \frac{EG \cdot HC}{PH} \\ &= GK_2 \end{aligned}$$

Recall that we defined K_1 and K_2 to be on GD. So K_1 and K_2 are the same distance from G and are both on GD on the same side as D.
$\therefore K_1 = K_2$.
So now the intersection point (K_1) of GD and FK is the same point as the intersection point (K_2) of GD and EK.
But FK and EK intersect at K and so, in fact, $K_1 = K_2 = K$.
So now we have that K lies on GD the perpendicular bisector of BC.
$\therefore KB = KC$, as required.

7. Merlin summons the n knights of Camelot for a conference. Each day, he assigns them to the n seats at the Round Table. From the second day on, any two neighbours may interchange their seats if they were not neighbours on the first day. The knights try to sit in some cyclic order which has already occurred before on an earlier day. If they succeed, then the conference comes to an end when the day is over.

   What is the maximum number of days for which Merlin can guarantee that the conference will last? (12 points)