The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Junior Paper: Years 8, 9, 10
Northern Autumn 2011 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. An integer $n > 1$ is written on the board. Alex replaces it by $n + d$ or $n - d$, where d is any divisor of n greater than 1. This is repeated with the new value of n.
   Is it possible for Alex to write 2011 on the board, at some point, regardless of the initial value of n? (3 points)

2. In $\triangle ABC$, P is a point on side AB such that $AP = 2PB$.
   If Q is the midpoint of AC and $CP = 2PQ$, prove $\triangle ABC$ is right-angled. (4 points)

3. A set of at least two objects, whose masses are all different, has the property that for any pair of objects from the set, there is a subset of the remaining masses whose total mass equals that of the chosen pair.
   What is the minimum number of objects in this set? (5 points)

4. A game is played on a 2012 row by k column board, where $k > 2$. A marker is placed in one of the cells of the left-most column. Two players move the marker in turn. During each move, the player whose turn it is, moves the marker one cell to the right, or one cell up or down to a cell that has not been occupied by the marker before. The game is over when either player moves the marker to the right-most column. There are two versions of the game. In Version A, the player who gets the marker to the right-most column wins, and in Version B, this player loses. However, only when the marker reaches the second-last column, do the players learn which version of the game they are playing.
   Does either player have a winning strategy? (6 points)

5. Let $a, b, c, d \in \mathbb{R}$ such that $0 < a, b, c, d < 1$ and $abcd = (1 - a)(1 - b)(1 - c)(1 - d)$.
   Prove that $(a + b + c + d) - (a + c)(b + d) \geq 1$. (6 points)

6. A car travels along a straight highway at 60 km/h. A 100 m long fence is standing parallel to the highway. Each second, the car's passenger measures the angle of vision of the fence.
   Prove that the sum of all angles measured is less than 1100°. (7 points)

7. Each vertex of a regular 45-gon is red, yellow or green, and there are 15 vertices of each colour.
   Prove that we can choose three vertices of each colour so that the three triangles formed by the chosen vertices of the same colour are congruent to one another. (9 points)