The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Some Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2011 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. An integer $n > 1$ is written on the board. Alex replaces it by $n + d$ or $n - d$, where d is any divisor of n greater than 1. This is repeated with the new value of n.
Is it possible for Alex to write 2011 on the board, at some point, regardless of the initial value of n? (3 points)

Solution. We start with $n > 1$ (given) and observe $n \mid n$.
So we can write $n - n = 0$ on the board at the first step.
Now $2011 \mid 0$. So we can write $2011 + 0 = 2011$ at the second step.
So, yes, it is possible for Alex to write 2011 on the board, at some point, regardless of the initial value of n.

Alternative solution. It seems that the setters really intended that no intermediate value of n could be 0. Assuming this, we can still do the following.
Let the initial value of n be n_0.
For the first 2010 steps, we add n_0, which is allowed since at each such step $n_0 \mid n$, i.e. inductively,

$$\begin{aligned}
\text{Step } 1 : n = n_0 &\mapsto n_0 + n_0 = 2n_0\\
\text{Step } 2 : n = 2n_0 &\mapsto 2n_0 + n_0 = 3n_0\\
&\vdots\\
\text{Step } 2010 : n = 2010n_0 &\mapsto 2010n_0 + n_0 = 2011n_0.
\end{aligned}$$

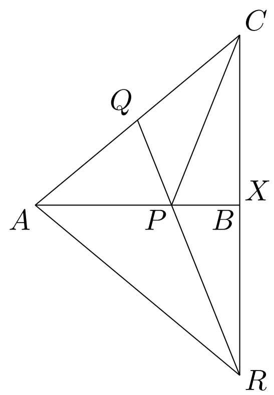
Now for the next $n_0 - 2$ steps, we subtract 2011, which is allowed since at each such step $2011 \mid n$, i.e. inductively,

$$\begin{aligned}
\text{Step } 2010+1 : n = 2011n_0 &\mapsto 2011n_0 - 2011 = 2011(n_0 - 1)\\
\text{Step } 2010+2 : n = 2011(n_0 - 1) &\mapsto 2011(n_0 - 1) - 2011 = 2011(n_0 - 2)\\
&\vdots\\
\text{Step } 2010 + n_0 - 2 : n = 2011\big(n_0 - (n_0 - 2)\big) &\mapsto 2011\big(n_0 - (n_0 - 2)\big) - 2011 = 2011.
\end{aligned}$$

Thus, after $2010 + n_0 - 2$ steps, Alex can write 2011, regardless of the initial value of n.

2. In $\triangle ABC$, P is a point on side AB such that $AP = 2PB$.
If Q is the midpoint of AC and $CP = 2PQ$, prove $\triangle ABC$ is right-angled. (4 points)

Solution. First, extend QP to R, such that $PR = 2PQ$.
Then RQ is a median of $\triangle ACR$, with P at the point of trisection of QR, so that P is the centroid of $\triangle ACR$.
Hence, AP is a median for $\triangle ACR$.
Let X be the intersection point of AP and CR.
Then $AP = 2PX$.
But B is such that $AP = 2PB$, and so $B = X$.



Now,

$$
\begin{aligned}
PR &= 2PQ && \\
&= PC && \text{(given)} \\
RB &= CR && (AB \text{ is a median of } \triangle ACR) \\
PB & \text{ is common} && \\
\therefore \triangle PBR &\cong \triangle PBC, && \text{by the SSS Rule} \\
\therefore \angle PBR &= \angle PBC = \angle ABC = 90^\circ && \text{(half of a straight angle)} \\
\therefore \triangle ABC & \text{ is right-angled (at } \angle B)
\end{aligned}
$$

3. A set of at least two objects, whose masses are all different, has the property that for any pair of objects from the set, there is a subset of the remaining masses whose total mass equals that of the chosen pair.
   What is the minimum number of objects in this set? (5 points)

   Solution. Let n be the minimum number of masses. Firstly, any pair of masses has mass greater than 0, so that there must be at least one more object. Hence, $n \geq 3$.
   Suppose $n = 3$. Without loss of generality, let the masses be x, y, z with $x < y < z$. Then taking the pair y, z, the property implies $x = y + z$. But $x < y < y + z\lightning$. So $n > 3$.
   Suppose $n = 4$. Without loss of generality, let the masses be w, x, y, z with $w < x < y < z$. Taking the pair y, z, we observe that their mass is larger than the remaining objects, i.e. $y + z > w + x$, and so again the property cannot be satisfied. So $n > 4$.
   Suppose $n = 5$. Without loss of generality, let the masses be v, w, x, y, z with $v < w < x < y < z$. Now,

   $$y + z > w + x > v + x > v + w,$$

   so to satisfy the property, we must have $y + z = v + w + x$. Since $z + x > y + w$, we must also have

   $$z + x = y + w + v > x + w + v = y + z,$$

   which implies $x > y\lightning$. So $n > 5$.
   On the other hand, with the six masses: $3, 4, 5, 6, 7, 8$ we have:

   $$
   \begin{aligned}
   &8 + 7 = 6 + 5 + 4, && 7 + 3 = 6 + 4 \\
   &8 + 6 = 7 + 4 + 3, && 6 + 3 = 5 + 4 \\
   &8 + 5 = 7 + 6, && 5 + 3 = 8 \\
   &8 + 4 = 7 + 5, && 4 + 3 = 7, \\
   &8 + 3 = 7 + 4 = 6 + 5, &&
   \end{aligned}
   $$

   which shows that the property is satisfied.
   Hence the minimum number n of masses is 6.

   Note. The masses $1, 2, 3, 4, 5, 6$ don't work!

4. A game is played on a 2012 row by k column board, where $k > 2$. A marker is placed in one of the cells of the left-most column. Two players move the marker in turn. During each move, the player whose turn it is, moves the marker one cell to the right, or one cell up or down to a cell that has not been occupied by the marker before. The game is over when either player moves the marker to the right-most column. There are two versions of the game. In Version A, the player who gets the marker to the right-most column wins, and in Version B, this player loses. However, only when the marker reaches the second-last column, do the players learn which version of the game they are playing.
   Does either player have a winning strategy? (6 points)

   Solution. Yes. Player 1 has a winning strategy:
   In the first $k - 2$ columns, Player 1 moves the marker up or down according to whether

there are an odd or even number of cells in that direction. Player 2 must move the marker vertically in the same direction as Player 1, in order not to place the marker on a previously occupied cell, or move the marker right to the next column, which s/he must inevitably do when Player 1 reaches the top or bottom cell of the column.
When the version is announced, Player 1 immediately moves to the rightmost column if Version A is announced, or continues with the previous strategy if Version B is announced.

5. Let $a, b, c, d \in \mathbb{R}$ such that $0 < a, b, c, d < 1$ and $abcd = (1-a)(1-b)(1-c)(1-d)$. Prove that $(a+b+c+d) - (a+c)(b+d) \geq 1$. (6 points)

Solution. Let $a, b, c, d \in \mathbb{R}$ such that

$$0 < a, b, c, d < 1 \tag{1}$$
$$\text{and } abcd = (1-a)(1-b)(1-c)(1-d). \tag{2}$$

Let $x = a + c$ and $y = b + d$. Then the inequality we are required to prove,

$$(a+b+c+d) - (a+c)(b+d) \geq 1 \tag{3}$$

is equivalent to

$$(x-1)(1-y) = (x+y) - xy - 1 \geq 0. \tag{4}$$

For a contradiction, suppose that (4) is false, i.e.

$$\begin{aligned} &(x-1)(1-y) < 0 \\ \iff &(x-1)(y-1) > 0 \\ \iff &x, y > 1 \text{ or } x, y < 1. \end{aligned}$$

Suppose that $x, y > 1$. Then

$$x = a + c > 1 \implies \begin{cases} a > 1 - c & \text{(a)} \\ c > 1 - a & \text{(b)} \end{cases}$$

$$y = b + d > 1 \implies \begin{cases} b > 1 - d & \text{(c)} \\ d > 1 - b & \text{(d)} \end{cases}$$

We note that the condition (1) ensures that

$$a, b, c, d, 1-a, 1-b, 1-c, 1-d > 0,$$

i.e. each side of the inequalities (a)–(d) is positive. Thus, we may multiply inequalities (a)–(d) and obtain

$$abcd > (1-a)(1-b)(1-c)(1-d)$$

which contradicts (2).
Now suppose $x, y < 1$. Then we obtain inequalities just like (a)–(d), but with each "$>$" replaced by "$<$", so that multiplying these inequalities we obtain

$$abcd < (1-a)(1-b)(1-c)(1-d)$$

which again contradicts (2).
Thus, either way we have a contradiction, and so in fact

$$(x-1)(1-y) \geq 0,$$

i.e. $a + b + c + d - (a+c)(b+d) \geq 1$.

6. A car travels along a straight highway at 60 km/h. A 100 m long fence is standing parallel to the highway. Each second, the car's passenger measures the angle of vision of the fence. Prove that the sum of all angles measured is less than $1100°$. (7 points)

7. Each vertex of a regular 45-gon is red, yellow or green, and there are 15 vertices of each colour.
Prove that we can choose three vertices of each colour so that the three triangles formed by the chosen vertices of the same colour are congruent to one another. (9 points)