The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems
Junior Paper: Years 8, 9, 10
Northern Autumn 2012 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. The decimal representation of $N \in \mathbb{N}$ uses only two different digits. N is at least 10 digits long, and adjacent digits are distinct.

   What is the greatest power of two that can divide N? (4 points)

2. Chip and Dale play a game. To start, Chip puts 222 nuts into 2 piles.
   Dale knows the way they have been divided and chooses an integer $N \in \{1, \dots, 222\}$.
   Then Chip moves, if necessary, one or more nuts to make a third pile such that one pile or a pair of piles contains a total of exactly N nuts.
   Dale then gets the nuts moved to the third pile by Chip.

   What is the largest number of nuts that Dale can get for sure, no matter how Chip acts? (5 points)

3. Some cells of an 11×11 table are filled out with a plus sign, such that the total number of pluses in the table and in any of its (contiguous) 2×2 sub-tables is even.

   Prove the number of pluses on the main diagonal of the table is also even. (6 points)

4. Let $\triangle ABC$, with incentre I, be such that X, Y, Z are the incentres of $\triangle AIB$, $\triangle BIC$ and $\triangle AIC$, respectively, and such that the incentre of $\triangle XYZ$ coincides with I.

   Is it necessarily true that $\triangle ABC$ is equilateral? (7 points)

5. A car is driven clockwise around a circular track. At noon Peter and Paul took up different positions on the track. The car passed each of them 30 times. Peter observed that each successive lap by the car was 1 second faster than the previous lap, while Paul observed that each successive lap was 1 second slower than the previous lap.

   Prove that Peter and Paul were observing for at least an hour and a half. (8 points)

6. (a) A point A is chosen inside a circle. Two perpendicular lines drawn through A intersect the circle at four points.

   Prove the centre of mass of the 4 points on the circle does not depend on the choice of the 2 lines. (4 points)

   (b) A regular $2n$-gon ($n \geq 2$) with centre A is drawn inside a circle (A does not necessarily coincide with the centre of the circle). The rays going from A through the vertices of the $2n$-gon meet the circle at $2n$ points. Then the $2n$-gon is rotated about A. The rays going from A through the new locations of the $2n$-gon vertices meet the circle at $2n$ new points. Let O and N be the centres of mass of old and new points on the circle, respectively.
   Prove that $O = N$. (4 points)

7. Peter and Paul play a game. First, Peter chooses $a \in \mathbb{N}$ such that the sum of its digits $S(a)$ is 2012. Paul wants to determine a; at first, he knows only that $S(a) = 2012$. On each turn, Paul chooses an $x \in \mathbb{N}$ and Peter responds with $S(|x - a|)$.

   What is the least number of turns, in which Paul can determine a for sure? (10 points)