The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Some Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2012 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. The decimal representation of $N \in \mathbb{N}$ uses only two different digits. N is at least 10 digits long, and adjacent digits are distinct.

   What is the greatest power of two that can divide N? (4 points)

   Solution. First suppose N has an even number of digits. Then its digit representation is $(abab\dots ab)$ for some $a, b \in \{0, 1, \dots, 9\}$.
   $\therefore\ N = m \cdot 1010\dots01$ for some $m = (ab)$.
   But $1010\dots01$ is odd. So the highest power of 2 that can divide N, is the highest power of 2 that can divide $m = (ab)$.
   Now, $2^6 = 64$ and $2^7 = 128$. So, the highest power of 2 that can divide $m = (ab)$ and hence N, is $2^6 = 64$, precisely when $a = 6$ and $b = 4$ (since any non-trivial multiple of 64 has more than 2 digits).
   Now suppose N has an odd number of digits. Then $N \geq 10^{10}$, since N has at least 10 digits. For a contradiction, suppose $2^7 \mid N$. Since the conditions require that the digits alternate, we have

   $$\begin{aligned} N &= (babab\dots ab) \text{ for some } a, b \in \{0, 1, \dots, 9\} \\ &= b \cdot 10^k + (abab\dots ab) \text{ for some } k \geq 10 \end{aligned}$$

   But $2^7 \,|\, 10^7 \,|\, 10^k$. So we must have

   $$2^7 \,|\, (abab\dots ab),$$

   which is a contradiction, since we showed above 2^6 is the highest power of 2 that can divide $(abab\dots ab)$.
   Thus whether N has an odd or even number of digits, $2^6 \parallel N$, which is achieved for example if $N = 6464646464$.

2. Chip and Dale play a game. To start, Chip puts 222 nuts into 2 piles.
   Dale knows the way they have been divided and chooses an integer $N \in \{1, \dots, 222\}$.
   Then Chip moves, if necessary, one or more nuts to make a third pile such that one pile or a pair of piles contains a total of exactly N nuts.
   Dale then gets the nuts moved to the third pile by Chip.

   What is the largest number of nuts that Dale can get for sure, no matter how Chip acts? (5 points)

   Solution. Let Chip's initial two piles be A and B, with a and b nuts, respectively, and assume without loss of generality, $a \leq b$. Make marks on the number line at $0, a, b, a + b = 222$. The mark at 0 corresponds to the third pile, which we label O, which initially has no nuts. After Dale chooses N, in order to move as few nuts as possible, Chip will choose to move nuts from pile or piles to adjust the closest mark to be N. If Chip chooses the piles to make $0, a, b, a + b = 222$ as evenly spaced as possible, which is when $b = 2a$ and hence $222 = 3a$, i.e. $a = 74$, $b = 148$, then N is at most a distance 37 from the closest of $0, a, b, a + b$, and so with this strategy Chip loses at most 37 nuts. Of course, if Chip has chosen the piles according to this strategy, then by choosing $N = 37$, Dale guarantees getting 37 nuts.

Now suppose Chip chooses a, b differently. If $a \le 74$ then $b \ge 148$ then by choosing $N = 111$, Dale guarantees getting ≥ 37 nuts. If $74 < a \le 111 \le b$, then by choosing $N = 37$ Dale is again guaranteed of getting 37 nuts.

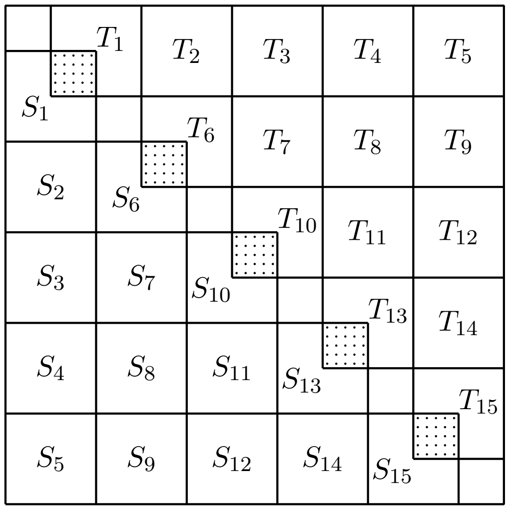
Thus 37 is the largest number of nuts that Dale can get for sure, no matter how Chip acts.

3. Some cells of an 11×11 table are filled out with a plus sign, such that the total number of pluses in the table and in any of its (contiguous) 2×2 sub-tables is even.

   Prove the number of pluses on the main diagonal of the table is also even. (6 points)

   Solution. Let G be the full 11×11 table, and let the S_i and T_i be the 2×2 sub-tables shown where S_j and T_j intersect in a shaded cell, for $j \in \{1, 6, 10, 13, 15\}$.
   Further let

$$
\begin{aligned}
S &= \bigcup_i S_i \\
T &= \bigcup_i T_i \\
S \,\Delta\, T &= (S \cup T) \setminus (S \cap T) \\
D &= G \setminus (S \,\Delta\, T).
\end{aligned}
$$



   Also, define a function p on the cells c of G, by

$$
p(c) = \begin{cases} 1, & \text{if there is a + in } c \\ 0, & \text{if there is not a + in } c. \end{cases}
$$

   Then since each 2×2 sub-table has an even number of pluses,

$$
\sum_{c \in S_i} p(c) \equiv 0 \pmod 2 \quad \text{and} \quad \sum_{c \in T_i} p(c) \equiv 0 \pmod 2, \text{ for all } i,
$$

   and since the S_i are mutually disjoint, and the T_i are mutually disjoint,

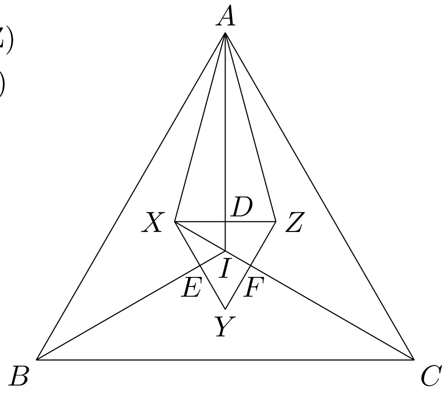
$$
\begin{aligned}
\sum_{c \in S} p(c) \equiv 0 \pmod 2 \quad &\text{and} \quad \sum_{c \in T} p(c) \equiv 0 \pmod 2 \\
\therefore 0 &\equiv \sum_{c \in S} p(c) + \sum_{c \in T} p(c) \pmod 2 \\
&\equiv \sum_{c \in S \Delta T} p(c) + 2 \sum_{c \in S \cap T} p(c) \pmod 2 \\
&\equiv \sum_{c \in S \Delta T} p(c) \pmod 2 \\
\therefore \sum_{c \in D} p(c) &\equiv \sum_{c \in G} p(c) - \sum_{c \in S \Delta T} p(c) \pmod 2 \\
&\equiv 0 - 0 \pmod 2 \\
&\equiv 0 \pmod 2.
\end{aligned}
$$

   Thus the number of pluses along the main diagonal is even.

4. Let $\triangle ABC$, with incentre I, be such that X, Y, Z are the incentres of $\triangle AIB$, $\triangle BIC$ and $\triangle AIC$, respectively, and such that the incentre of $\triangle XYZ$ coincides with I.

   Is it necessarily true that $\triangle ABC$ is equilateral? (7 points)

Solution. Let $D = AI \cap XZ$, $E = AI \cap XY$, $F = CI \cap YZ$.



$$
\begin{aligned}
\angle EXI &= \angle DXI, &&\text{since } I = \text{incentre(XYZ)}\\
\angle EIX &= \angle DIX, &&\text{since } I = \text{incentre(ABI)}\\
IX &\text{ is common}\\
\therefore \triangle EIX &\cong \triangle DIX, &&\text{by the AAS Rule}\\
\therefore \angle IEX &= \angle IDX = \theta \text{ say, and } XE = XD\\
\text{Similarly, } \angle IDZ &= \angle IFZ = \phi \text{ say, and } DZ = FZ,\\
\angle IEY &= \angle IFY = \psi \text{ say, and } EY = FY\\
\text{But } \theta + \phi &= \theta + \psi = \phi + \psi = 180^\circ\\
\therefore\ \theta &= \phi = \psi = 90^\circ.
\end{aligned}
$$

$$
\begin{aligned}
\text{Now, } \angle BAX &= \angle XAI = \alpha \text{ say,} &&\text{since } X = \text{incentre}(AIB)\\
\therefore 2\alpha = \angle BAI &= \angle IAC, &&\text{since } I = \text{incentre}(ABC)\\
&= 2\angle ZAI, &&\text{since } Z = \text{incentre}(AIC)\\
\therefore \angle XAD = \angle XAI &= \alpha = \angle ZAI = \angle ZAD\\
\angle ADX &= \angle ADZ = 90^\circ &&(\text{vertically opposite } \angle IDZ = \angle IDX)\\
AD &\text{ is common}\\
\therefore \triangle XAD &\cong \triangle ZAD, &&\text{by the AAS Rule}\\
\therefore DX &= DY, \text{ i.e. } DX = \tfrac{1}{2}XZ\\
\text{Similarly, } EX &= \tfrac{1}{2}XY \text{ and } FZ = \tfrac{1}{2}YZ\\
\therefore \tfrac{1}{2}XY = EX &= DX = \tfrac{1}{2}XZ\\
&= FZ = \tfrac{1}{2}YZ\\
\therefore XY &= XZ = YZ\\
\therefore \triangle XYZ &\text{ is equilateral}\\
\text{Also, } XEID &\text{ is cyclic,} &&\text{since } \angle E + \angle D = 180^\circ\\
\therefore \angle BIA = \angle EID &= 180^\circ - \angle EXD = 120^\circ\\
\therefore \angle BIA + \angle ABI &= 60^\circ\\
\text{Similarly, } \angle CAI + \angle ACI &= 60^\circ,\\
\angle BCI + \angle CBI &= 60^\circ
\end{aligned}
$$

i.e. the sum of any two of $\frac{1}{2}\angle BAC, \frac{1}{2}\angle ACB, \frac{1}{2}\angle CBA$ is 60°

$$
\begin{aligned}
\therefore \tfrac{1}{2}\angle BAC = \tfrac{1}{2}\angle ACB &= \tfrac{1}{2}\angle CBA = 30^\circ\\
\therefore \angle BAC = \angle ACB &= \angle CBA = 60^\circ\\
\therefore \triangle ABC &\text{ is equilateral.}
\end{aligned}
$$

Several times above we used the idea:
Lemma If the sum of any two of λ, μ, ν is K then $\lambda = \mu = \nu = K/2$.
Proof. We have

$$
\begin{aligned}
K &= \lambda + \mu = \lambda + \nu\\
\therefore \mu &= \nu,
\end{aligned}
$$

and similarly, $\nu = \lambda$. Substituting, back in the first equation, we have

$$
\begin{aligned}
K &= 2\nu\\
\therefore \lambda &= \mu = \nu = K/2.
\end{aligned}
$$

□

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5. A car is driven clockwise around a circular track. At noon Peter and Paul took up different positions on the track. The car passed each of them 30 times. Peter observed that each successive lap by the car was 1 second faster than the previous lap, while Paul observed that each successive lap was 1 second slower than the previous lap.

   Prove that Peter and Paul were observing for at least an hour and a half. (8 points)

6. (a) A point A is chosen inside a circle. Two perpendicular lines drawn through A intersect the circle at four points.

   Prove the centre of mass of the 4 points on the circle does not depend on the choice of the 2 lines. (4 points)

   (b) A regular $2n$-gon ($n \geq 2$) with centre A is drawn inside a circle (A does not necessarily coincide with the centre of the circle). The rays going from A through the vertices of the $2n$-gon meet the circle at $2n$ points. Then the $2n$-gon is rotated about A. The rays going from A through the new locations of the $2n$-gon vertices meet the circle at $2n$ new points. Let O and N be the centres of mass of old and new points on the circle, respectively.

   Prove that $O = N$. (4 points)

7. Peter and Paul play a game. First, Peter chooses $a \in \mathbb{N}$ such that the sum of its digits $S(a)$ is 2012. Paul wants to determine a; at first, he knows only that $S(a) = 2012$. On each turn, Paul chooses an $x \in \mathbb{N}$ and Peter responds with $S(|x - a|)$.

   What is the least number of turns, in which Paul can determine a for sure? (10 points)