The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

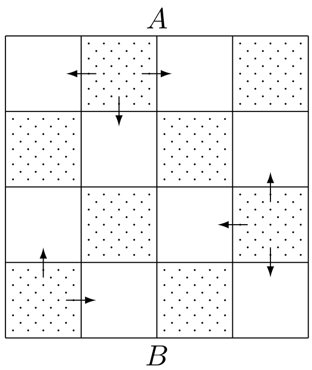
AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2000 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Each of the 16 squares in a 4×4 table contains a number. For any square, the sum of the numbers in the squares sharing a common side with the chosen square is equal to 1. Determine the sum of all 16 numbers in the table. (R Zhenodarov, 3 points)

   Solution. In the diagram, below left, three shaded squares are shown with arrows emanating from them. The arrows indicate the neighbours of these shaded squares sharing an edge with the shaded square. Each white square is a neighbour of exactly one of these three shaded squares. Hence, by the given information, the total of the numbers in the white squares is 3.

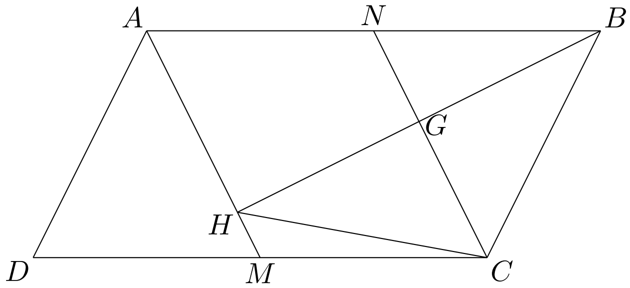


$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	0	0	$\frac{1}{2}$
$\frac{1}{2}$	0	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

   Now reflect the left diagram in the axis AB. This takes each shaded square to a white square and each white square to a shaded square. Using the same argument as before with respect to where the arrows now lie, we see that the total of the numbers in the shaded squares is also 3.
   So, if there is a configuration of numbers in the squares that fit the given conditions, then its total must be 6.
   The right diagram shows that it is possible to find a configuration of numbers that fit the conditions. By symmetry, it is sufficient to check a corner square (which is adjacent to two centre-edge squares of total value $\frac{1}{2} + \frac{1}{2} = 1$), a centre-edge square (which is adjacent to two edge-squares and a middle square of total value $\frac{1}{2} + \frac{1}{2} + 0 = 1$) and a middle square (which is adjacent to two edge squares and two middle squares of total value $\frac{1}{2} + \frac{1}{2} + 0 + 0 = 1$).

2. Given that $ABCD$ is a parallelogram, M is the midpoint of side CD and H is the foot of the perpendicular from B to line AM, prove that $\triangle BCH$ is isosceles.
   (M Volchkevich, 3 points)

   Solution. We are given parallelogram $ABCD$, with M the midpoint of side CD and H the foot of the perpendicular from B to line AM. Further, we construct N as the midpoint of AB and construct CN to intersect BH at G.



Now $MC \parallel AN$ and $MC = \frac{1}{2}CD = \frac{1}{2}AB = AN$. $\therefore AMCN$ is a parallelogram.
$\therefore AM \parallel NC$.
Since parallel lines cut transversals in equal proportions, and $BN = NA$, it follows that $BG = GH = HG$.
Also, $\angle BHM = \angle BGC = 90°$ (corresponding angles). Thus we have

$$\begin{aligned} BG &= HG \\ \angle BGC &= \angle HGC = 90°, && \text{(complements of a straight angle)} \\ GC &\text{ common} \\ \therefore \triangle CBG &\cong \triangle CHG, && \text{(by the SAS Rule)} \\ \therefore CB &= CH \end{aligned}$$

and so $\triangle BCH$ is isosceles.

3. (a) On a blackboard are written 100 different numbers. Prove that you can choose 8 of them so that their average value is not equal to that of any 9 of the numbers on the blackboard. (2 points)

   Solution. Since the numbers are distinct, we may choose the smallest 8 numbers. Then their average is smaller (and hence not equal) to the average of any 9 of the numbers.
   Alternatively, we may choose the largest 8 numbers, and similarly their average is larger (and hence not equal) to the average of any 9 of the numbers.

   (b) On a blackboard are written 100 integers. For any 8 of them, you can find 9 numbers on the blackboard so that the average value of the 8 numbers is equal to that of the 9. Prove that all the numbers on the blackboard are equal.
   (A Shapovalov, 4 points)

   Solution. Here's a start to this problem.
   Let the 100 integers be $x_1, x_2, \ldots, x_{100}$ and assume without loss of generality that they are ordered such that
   $$x_1 \leq x_2 \leq \cdots \leq x_{100}.$$

4. Among a set of 32 coins, all identical in appearance, 30 are real and 2 are fake. Any two real coins have the same weight. The fake coins have the same weight, which is different from the weight of a real coin. How can one divide the coins into two groups of equal total weight by using a balance at most 4 times? (A Shapovalov, 5 points)