The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Tournament of the Towns Problems
Junior Paper: Years 8, 9, 10
Northern Autumn 2006 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Two positive integers are written on the blackboard. Mary records in her notebook the square of the smaller number and replaces the larger number on the board by the difference of the two numbers. With the new pair of numbers, she repeats the process, and continues until one of the numbers on the blackboard becomes zero. What will be the sum of the numbers in Mary's notebook at that point? (4 points)

2. A Knight always tells the truth. A Knave always lies. A Normal may either lie or tell the truth. You are allowed to ask questions that can be answered with "yes" or "no", such as "Is a person a Normal?"

   (a) There are three people in front of you. One is a Knight, another one is a Knave, and the third one is a Normal. They all know the identities of one another. How too can you learn the identity of each? (1 point)

   (b) There are four people in front of you. One is a Knight, another one is a Knave, and the other two are Normals. They all know the identities of one another. Prove that the Normals may agree in advance to answer your questions in such a way that you will not be able to learn the identities of any of the four people. (3 points)

3. (a) Prove that from 2007 given positive integers, one of them can be chosen so that the product of the remaining numbers is expressible in the form $a^2 - b^2$ for some positive integers a and b. (2 points)

   (b) One of 2007 given positive integers is 2006. Prove that if there is a unique number among them such that the product of the remaining numbers is expressible in the form $a^2 - b^2$ for some positive integers a and b, then this unique number is 2006. (2 points)

4. Given triangle ABC, BC is extended beyond B to the point D such that $BD = BA$. The bisectors of the exterior angles at vertices B and C intersect at the point M. Prove that A, D, M and C are concyclic. (4 points)

5. A square is dissected into n congruent non-convex polygons whose sides are parallel to the sides of the square, and no two of the polygons are parallel translates of each other. What is the maximum value of n? (4 points)