The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

**AMO TRAINING SESSIONS**

# Tournament of the Towns Problems with Solutions
# Junior Paper: Years 8, 9, 10
# Northern Autumn 2006 (O Level)

**Note:** Each contestant is credited with the largest sum of points obtained for three problems.

1. Two positive integers are written on the blackboard. Mary records in her notebook the square of the smaller number and replaces the larger number on the board by the difference of the two numbers. With the new pair of numbers, she repeats the process, and continues until one of the numbers on the blackboard becomes zero. What will be the sum of the numbers in Mary's notebook at that point? (4 points)

   **Solution.** We claim that the sum of the numbers in Mary's notebook is equal to the product of the two numbers originally on the blackboard. We use induction on the number $n$ of steps for Mary to reduce one of the numbers to 0. For $n = 1$, the two numbers on the blackboard must be equal to each other. In recording the square of the smaller number, Mary is in fact recording the product of the two numbers. Suppose the claim holds for some $n \geq 1$. Let the original numbers be $x$ and $y$ with $x < y$. Then Mary records $x^2$ in her notebook and replaces $y$ by $y - x$. By the induction hypothesis, the sum of the remaining numbers in her notebook is equal to $x(y - x)$, so that the sum of all the numbers in her notebook is equal to $x^2 + x(y - x) = xy$.

2. A Knight always tells the truth. A Knave always lies. A Normal may either lie or tell the truth. You are allowed to ask questions that can be answered with "yes" or "no", such as "Is a person a Normal?"

   (a) There are three people in front of you. One is a Knight, another one is a Knave, and the third one is a Normal. They all know the identities of one another. How too can you learn the identity of each? (1 point)

   **Solution.** Ask each of the three people: "Are you a Normal?" The Knight, telling the truth, will answer "No", and the Knave, lying will answer "Yes". The Normal will either answer "Yes" or "No".
   If there are two "Yes" answers, then we know that the "No" came from the Knight, and we then pick one of the "Yes" respondents and ask the Knight if the chosen "Yes" respondent is a Normal, and either way we know the type of all three.
   Similarly, if there are two "No" answers we know that the "Yes" came from the Knave, and we can pick one of the "No" respondents and ask the Knave if the chosen "No" respondent is a Normal. Whatever the Knave says, we know the opposite is true, and again we will know the type of all three.

(b) There are four people in front of you. One is a Knight, another one is a Knave, and the other two are Normals. They all know the identities of one another. Prove that the Normals may agree in advance to answer your questions in such a way that you will not be able to learn the identities of any of the four people. (3 points)

**Solution.** The first Normal will act as though he is a Knight while the second Normal will act as though he is a Knave. Then we cannot tell the difference between the first Normal and the Knight, nor between the second Normal and the Knave.

3. (a) Prove that from 2007 given positive integers, one of them can be chosen so that the product of the remaining numbers is expressible in the form $a^2-b^2$ for some positive integers $a$ and $b$. (2 points)

(b) One of 2007 given positive integers is 2006. Prove that if there is a unique number among them such that the product of the remaining numbers is expressible in the form $a^2-b^2$ for some positive integers $a$ and $b$, then this unique number is 2006. (2 points)

**Solution.** Suppose a number is expressible in the form $a^2-b^2=(a+b)(a-b)$. If $a$ and $b$ are of the same parity, then the product is divisible by 4. If they are of opposite parity, then the product is odd. Conversely, a number of the form $4n$ may be expressed as $(n+1)^2-(n-1)^2$ while a number of the form $2n+1$ may be expressed as $(n+1)^2-n^2$. Hence a number is not expressible in the form $a^2-b^2$ if and only if it is of the form $4n+2$. The only way in which a product takes the form $4n+2$ is when exactly one of the factors is of that form, and the others are odd.

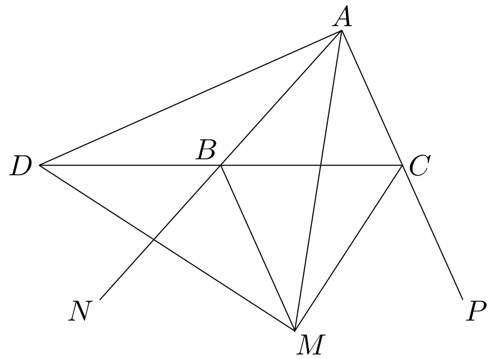
(a) Suppose an even number of the 2007 numbers is of the form $4n+2$. Then there exists at least one number not of this form, and we choose this number. Suppose an odd number of the 2007 numbers is of the form $4n+2$. Then we choose any of these. Among the emaining 2006 numbers, there will not be exactly one number of the form $4n+2$. Hence their product is expressible in the form $a^2-b^2$.

(b) If there is a number of the form $4n+2$ other than 2006, then any of the other 2005 numbers may be chosen so that the product of the remaining 2006 numbers will not be of the form $4n+2$. Hence the choice will not be unique. It follows that 2006 is the only number of the form $4n+2$, and it must be the chosen number.

4. Given triangle $ABC$, $BC$ is extended beyond $B$ to the point $D$ such that $BD = BA$. The bisectors of the exterior angles at vertices $B$ and $C$ intersect at the point $M$. Prove that $A$, $D$, $M$ and $C$ are concyclic. (4 points)

**Solution.** Produce $AB$ to $N$. Then $\angle NBC$ is the exterior angle at $B$. Produce $AC$ to $P$. Then $\angle PCB$ is the exterior angle at $C$. Let $M$ be the intersection point of the bisectors of $\angle NBC$ and $\angle PCB$.
Then,

$$\begin{aligned} \angle NBM &= \angle CBM \\ \angle DBN &= \angle ABC, && \text{(opposite angles)} \\ \therefore \angle DBM &= \angle ABM \end{aligned}$$




Now,

$$\begin{aligned} BD &= BA, && \text{(given)} \\ \angle DBM &= \angle ABM, && \text{(from above)} \\ BM & \text{ common} \\ \therefore \triangle DBM &\cong \triangle ABM, && \text{(by the SAS Rule)} \\ \therefore \angle MDC &= \angle MAN \end{aligned}$$

Now $M$ is an excentre of triangle $ABC$. Hence $\angle MAN = \angle MAC$. From $\angle MDC = \angle MAC$, we can conclude that $A$, $C$, $M$ and $D$ are concyclic.

5. A square is dissected into $n$ congruent non-convex polygons whose sides are parallel to the sides of the square, and no two of the polygons are parallel translates of each other. What is the maximum value of $n$? (4 points)

**Solution.**
The maximum value of $n$ is at most 8 because such a polygon can only have 8 possible orientations. We may use each of them once as otherwise we would have two copies which are parallel translates of each other. The maximum value is in fact 8 as it is attained by the polygon in the diagram shown at right.