The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2007 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Black and white checkers are placed on an 8×8 chessboard, with at most one checker on each cell. What is the maximum number of checkers that can be placed such that each row and each column contains twice as many white checkers as black ones? (3 points)

   Solution.
   For a single row the maximum number of black checkers is 2, since 3 or more black checkers in a row would imply 6 or more white checkers and hence at least 9 (> 8) checkers in the row, which is impossible since there are only 8 cells in each row. Thus each row can have at most 6 checkers (2 black and 4 white).
   So there can be at most $8 \times 6 = 48$ checkers in a configuration that satisfies the required conditions (6 checkers in each of the 8 rows). Indeed the following configuration shows that such a configuration with 48 checkers is possible. So 48 is indeed the maximum.

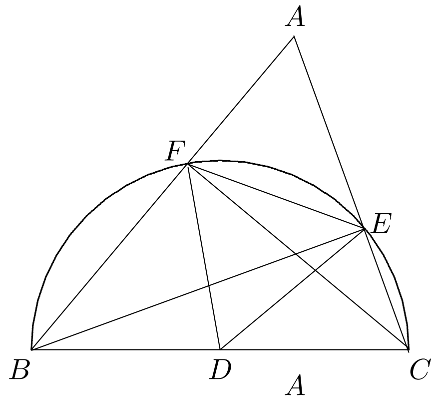
		●	●	○	○	○	○
		●	●	○	○	○	○
●	●			○	○	○	○
●	●			○	○	○	○
○	○	○	○			●	●
○	○	○	○			●	●
○	○	○	○	●	●		
○	○	○	○	●	●		

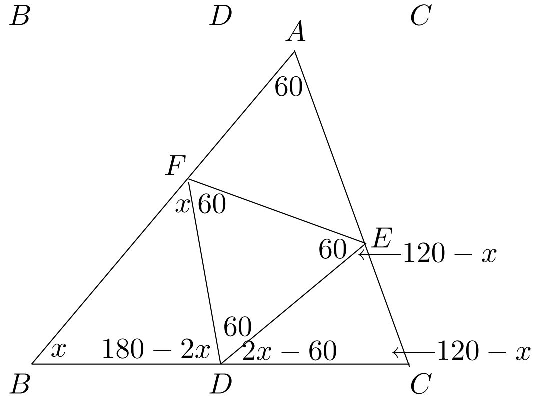
2. Initially, the number 1 and a non-integral number x are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or difference on the blackboard. We can also choose a non-zero number on the blackboard and write its reciprocal on the blackboard. Is it possible to write x^2 on the blackboard in a finite number of moves? (4 points)

   Solution. Yes, it is possible. Starting with 1 and x on the board,
   we can write their sum $1 + x$ and their difference $1 - x$.
   Next we can write the reciprocals $1/(1 + x)$ and $1/(1 - x)$,
   followed by their sum $1/(1 + x) + 1/(1 - x) = 2/(1 - x^2)$,
   followed by its reciprocal $(1 - x^2)/2$.
   Adding $(1 - x^2)/2$ to itself we obtain $1 - x^2$.
   Finally the difference of 1 and $1 - x^2$ is x^2.

3. D is the midpoint of the side BC of $\triangle ABC$. E and F are points on CA and AB respectively, such that $BE \perp CA$ and $CF \perp AB$. If DEF is an equilateral triangle, does it follow that $\triangle ABC$ is also equilateral? (4 points)

Solution.
First we recall that the locus of all points X such that $\angle BXC = 90°$ is a circle with diameter BC (and centre D).
Since $\angle BFC = \angle CEB = 90°$, both E and F must lie on a semicircle with diameter BC.
Thus, in particular, $DE = DF = DB = DC$ (radii of the semicircle $\overset{\frown}{BC}$).
Thus to construct the figure first draw the base BC with midpoint D and draw the semicircle $\overset{\frown}{BC}$.
Now choose a point F on $\overset{\frown}{BC}$, and with compass set to the radius DF of the semicircle, use the compass to mark off a point E at distance DF from F. In this way, we have constructed $\triangle DFE$ so that it is equilateral.
Now produce BF and CE to intersect at A.
All triangles meeting the required conditions may be constructed this way.
In particular, observe that we had a choice in positioning F, so that $\angle ABC = \angle FBC$ need not be $60°$. Hence $\triangle ABC$ need not be equilateral.





Further investigation
In the second diagram, we let $x = \angle FBD$, and using the facts that triangles DBF and DEC are isosceles, and that the sum of the angles of a triangle or straight line is $180°$ we obtain the angles (in degrees) shown. Now angles FDB and EDC are non-negative, so that

$$\begin{aligned} 180 - 2x \geq 0 &\implies x \leq 90 \\ 2x - 60 \geq 0 &\implies x \geq 30 \\ \therefore 30° \leq x &\leq 90° \end{aligned}$$

In drawing a 'generic' diagram, we chose $x = 50°$.

4. Each cell of a 29×29 table contains one of the integers $1, 2, 3, \ldots, 29$, and each of these integers appears 29 times. The sum of all the numbers above the main diagonal is equal to three times the sum of all the numbers below this diagonal. Determine the number in the central cell of the table. (5 points)

   Solution. There are 29 numbers on the main diagonal 29×14 numbers above the main diagonal and 29×14 numbers below the main diagonal.
   The sum of the largest 29×14 numbers is

   $$29(16 + 17 + \cdots + 29) = 29 \times \frac{14}{2} \times (16 + 29) = 29 \times 7 \times 45,$$

   while the sum of the smallest 29×14 numbers is

   $$29(1 + 2 + \cdots + 14) = 29 \times \frac{14}{2} \times (1 + 14) = 29 \times 7 \times 15.$$

   Since the sum of 29×14 can be no larger than the former sum and no smaller than the latter sum, the ratio of the sums of two different sets of 29×14 is as large as it can be with these two sets of numbers (any other choice of the two sets, necessarily has a smaller ratio). Since the ratio is exactly 3, the required ratio, the largest 29×14 numbers are all above the main diagonal and the smallest 29×14 numbers are all below the main diagonal, leaving the 29 15s to lie *on* the diagonal, i.e. *every* number on the main diagonal, including the central cell, is 15.

5. The audience chooses two of five cards, numbered from 1 to 5 respectively. The assistant of a magician chooses two of the remaining cards, and asks the audience to take them to the magician, who is in another room. The two cards are presented to the magician in arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only form the two cards he receives. Explain how this may be done. (5 points)

   Solution. Arrange the numbers in order 1 to 5 in a circle so that 1 may be considered to follow 5. One possible strategy depends on whether the audience selects consecutive cards or non-consecutive cards. If the audience selects two consecutive cards, the assistant chooses the next two consecutive cards, e.g. if the audience selects 3 and 4, the assistant chooses 5 and 1. If the audience selects non-consecutive cards, the assistant chooses the next card in sequence for each of the cards, e.g. if the audience selects 1 and 3, the assistant chooses 2 and 4. The magician can then reverse this strategy to deduce the cards, i.e. if he receives consecutive cards 5 and 1, he knows that the preceding consecutive cards 3 and 4 were selected by the audience, and if he receives non-consecutive cards 2 and 4, then he knows that the cards 1 and 3 preceding the individual cards were selected by the audience.