The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

**AMO/TT TRAINING SESSIONS**

# Tournament of the Towns Problems with Most Solutions
## Junior Paper: Years 8, 9, 10
## Northern Autumn 2008 (O Level)

**Note:** Each contestant is credited with the largest sum of points obtained for three problems.

1. Each of ten boxes contains a different number of pencils. No two pencils of the same colour are in any box. Prove that one can choose one pencil from each box so that all chosen pencils are of different colours. (3 points)

   **Solution.** Label the boxes from 1 to 10 in increasing order of the number of pencils inside. First we observe that the $n^{\text{th}}$ pencil box has at least $n$ pencils. We will prove the desired result by induction. Define

   $P(n)$ : One pencil from each of the first $n$ boxes can be chosen, such that they are all of different colours.

   From the first pencil box, we may choose any pencil in the box. So $P(1)$ holds.
   Now, assume $P(k)$ holds, i.e. that it is possible to choose one pencil from each of the first $k$ boxes, such that they are all of different colours, and that we have chosen $k$ coloured pencils in such a manner. Then the $(k+1)^{\text{st}}$ box has at least $k+1$ differently coloured pencils, at least one of which, is a different colour to the $k$ coloured pencils already chosen. Choosing such a differently coloured pencil, we have that $P(k+1)$ holds, if $P(k)$ does.
   Thus the induction is complete, and so $P(n)$ holds for all natural numbers $n$.
   In particular, $P(10)$ holds. Thus we can choose one pencil from each of the 10 boxes in such a way that all pencils are of different colours.

2. We are given fifty distinct positive integers of which twenty-five are not greater than 50. The others are greater than 50, but not greater than 100. No two of the given numbers differ by 50. Find the sum of the fifty numbers. (3 points)

   **Solution.** Since no two of the given numbers differ by 50, we can choose at most one number from each pair $\{k, 50+k\}$ for $1 \le k \le 50$.
   Since 50 numbers are chosen, we must choose *exactly* one number from each of these pairs. So the numbers are:

   $$1 + 50a_1, 2 + 50a_2, \dots, k + 50a_k, \dots, 50 + a_{50},$$

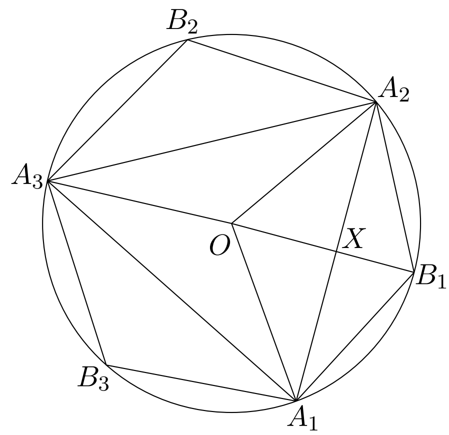
   where each $a_k \in \{0, 1\}$.
   Since 25 of the numbers are $\le 50$ and 25 are $> 50$, *exactly* 25 of the $a_k$ are 0, and 25 are 1.

   $$\begin{aligned}
   \therefore \text{The sum of the nos.} &= \sum_{k=1}^{50}(k + 50a_k)\\
   &= \sum_{k=1}^{50} k + 50\sum_{k=1}^{50} a_k\\
   &= \frac{50 \cdot 51}{2} + 50 \cdot 25\\
   &= 25(51 + 50) = 25 \cdot 101 = 2525.
   \end{aligned}$$

3. Let $A_1A_2A_3$ be an acute-angled triangle inscribed in a circle of radius 2. Prove that one can choose points $B_1$, $B_2$ and $B_3$ on the arcs $A_1A_2$, $A_2A_3$ and $A_3A_1$ respectively, such that the values of the area of the hexagon $A_1B_1A_2B_2A_3B_3$ and perimeter of $\triangle A_1A_2A_3$ are equal. (4 points)

**Solution.** Since $\triangle A_1A_2A_3$ is acute-angled, the centre $O$ of the circle lies inside the triangle. This is so since each angle $\angle A_iA_jA_k$ acute implies that double its value, the angle $\angle A_iOA_k$ is less than a straight angle.

Draw a radius from $O$ perpendicular to $A_1A_2$ and let the point where the radius meets the circle be $B_1$. Let $OB_1$ and $A_1A_2$ meet at $X$. Then $\triangle OA_1A_2$ and $\triangle B_1A_1A_2$ have common base $A_1A_2$ and corresponding perpendicular heights $OX$ and $B_1X$, respectively. Represent the area of a figure $UV\ldots Z$ by $(UV\ldots Z)$, then




$$
\begin{aligned}
(OA_1B_1A_2) &= (OA_1A_2) + (B_1A_1A_2)\\
&= \tfrac{1}{2}\cdot A_1A_2\cdot OX + \tfrac{1}{2}\cdot A_1A_2\cdot B_1X\\
&= \tfrac{1}{2}\cdot A_1A_2\cdot OB_1\\
&= \tfrac{1}{2}\cdot 2\cdot A_1A_2, \qquad \text{since radius } OB_1 = 2\\
&= A_1A_2
\end{aligned}
$$

Similarly, draw radii $OB_2 \perp A_2A_3$ and $OB_3 \perp A_3A_1$. Then the area of the hexagon,

$$
\begin{aligned}
(A_1B_1A_2B_2A_3B_3) &= (OA_1B_1A_2) + (OA_2B_2A_3) + (OA_3B_3A_1)\\
&= A_1A_2 + A_2A_3 + A_3A_1
\end{aligned}
$$

and hence for this choice of $B_1$, $B_2$ and $B_3$, the area of the hexagon $A_1B_1A_2B_2A_3B_3$ is numerically equal to the perimeter of $\triangle A_1A_2A_3$.

4. We are given three distinct positive integers, one of which is the average of the other two. Can the product of all three numbers be equal to the $2008^{\text{th}}$ power of some positive integer? (4 points)

**Solution.** Let the three positive integers be $x$, $2x$ and $3x$. Then the average of $x$ and $3x$ is $2x$, and their product is

$$x\cdot 2x\cdot 3x = 2\cdot 3x^3.$$

Observe that $2008 \equiv 1 \pmod 3$, and hence the product will be the $2008^{\text{th}}$ power of a positive integer if

$$x = 2^{(2008-1)/3}\cdot 3^{(2008-1)/3} = 6^{669},$$

i.e. the answer is: Yes, since, e.g. the three positive integers $6^{669}, 2\cdot 6^{669}, 3\cdot 6^{669}$ are such that the middle one is the average of the other two and their product is

$$2\cdot 3\cdot (6^{669})^3 = 6^{2008}$$

which is the $2008^{\text{th}}$ power of a positive integer.

5. Several athletes started running at one end of a straight track at the same time. Their speeds are different, but constant. When they reach the other end of the track, they turn around and run back to the start point. There they turn around again and run back to the other end of the track, and so on. Some time after the start, all athletes meet at the same point. Prove that such an event will happen again. (4 points)