The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

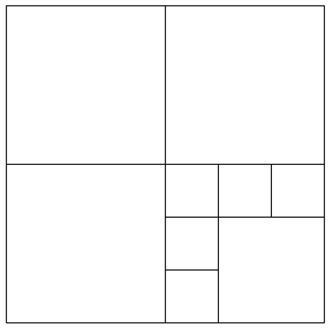
AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Brief Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2009 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Is it possible to cut a square into 9 smaller squares, so that one of the smaller squares is coloured white, three are coloured grey and the remaining five are coloured black, and such that the smaller squares of the same colour have the same size, while the smaller squares that are coloured differently have different size? (3 points)

   Solution. Yes, it is possible, as the following configuration demonstrates. There is one square of side length 2 units, three squares of side length 3 units, and five squares of side length 1 unit, making up a square of side length 6 units.



2. There are 40 weights with masses $1\,\text{g}, 2\,\text{g}, \ldots, 40\,\text{g}$, respectively. Ten weights with even masses are put on the left pan of some scales, and ten weights with odd masses are put on the right pan of the same scales, such that the scales are balanced.

   Prove that on one pan there are two weights whose masses differ by $20\,\text{g}$. (4 points)

   Solution. Organise the weights into pairs that differ in mass by $20\,\text{g}$.

Even pairs	Odd pairs
$2, 22$	$1, 21$
$4, 24$	$3, 23$
$\vdots$	$\vdots$
$20, 40$	$19, 39$

Suppose for a contradiction that the pans can be balanced as prescribed but no pan has two weights whose masses differ by $20\,\text{g}$.
Then each pan has exactly one of each pair as given above, and so working modulo 20, for balance, the sums of the left pan masses and right pan masses must satisfy,

$$\sum_{k=1}^{10} 2k \equiv \sum_{k=1}^{10} (2k-1) \pmod{20}.$$

But

$$
\begin{aligned}
\sum_{k=1}^{10} 2k - \sum_{k=1}^{10}(2k-1) &= \sum_{k=1}^{10} 1 \\
&= 10 \\
&\not\equiv 0 \pmod{20}.
\end{aligned}
$$

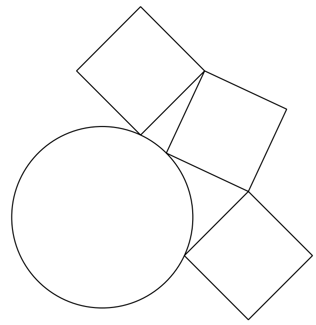
Thus the scales cannot be balanced, a contradiction.
So, in fact, there must be a pair of masses on a pan whose masses differ by 20 g.

3. To the circumference of a disk of radius 5 cm, squares of side length 5 cm are attached, one after another, while possible, such that:

   (i) one vertex of each square lies on the circumference of the disk;

   (ii) the squares do not overlap; and

   (iii) each square has a common vertex with its predecessor.

   Determine how many squares can be attached to the disk and prove that the first and last square attached will have a common vertex. (The first three squares of a possible arrangement are shown in the diagram.) (4 points)

   Solution. The essential idea is to first show that two vertices of each square subtend an angle of 45° at the centre, irrespective of orientation, which shows that $360/45 = 8$ squares can be fit round the disk.
   Let the distances of the two vertices of a square incident with adjacent squares be distances s and ℓ from the centre of the disk.
   Then the corresponding vertices of the next square are ℓ and s from the centre of the disk.
   Now for the eight squares, the sequence s, ℓ, ℓ, s occurs four times and the first and last squares have incident vertices a common distance s from the centre of the disk.

4. A 7-digit code, consisting of seven distinct digits, is called *good.* Suppose the password for a safe is a *good code,* and that the safe can be opened if an entered code is good and a digit of that code and the corresponding digit of the password are the same at some position.

   Is there a guaranteed method of opening the safe with fewer than 7 attempts without knowing the password? (5 points)

   Solution. Yes. We claim that one of the following six codes will open the safe:

   0 1 2 3 4 5 6
   1 2 3 4 5 0 7
   2 3 4 5 0 1 8
   3 4 5 0 1 2 9
   4 5 0 1 2 3 6
   5 0 1 2 3 4 7.

(Observe that the first six digits are $0, 1, 2, 3, 4, 5$ rotated, and that the last digit is merely chosen so that each code is *good.*)
Suppose, for a contradiction, that none of these codes open the safe.
Then each of the first six digits are not any of $0, 1, 2, 3, 4, 5$, and hence must each be members of $\{6, 7, 8, 9\}$.
But then by the Pigeon Hole Principle (at least) two of the first six digits of the safe's passcode must be the same.
$\therefore$ the safe's passcode is not *good,* which is a contradiction.
$\therefore$ one of the codes above opens the safe.

5. On a new website, 2000 people have registered. Each of them invited 1000 people (from the other people registered) to be their friends. Two people are regarded as friends if and only if each invited the other to be their friend.

   What is the least number of pairs of friends possible on the website? (5 points)

   Solution. Label the people $0, 1, 2, \dots, 1999 \pmod{2000}$.
   We note that if the person labelled k chooses to invite persons labelled $k+1, k+2, \dots, k+1000 \pmod{2000}$, for each k, then only the pairs $k, k+1000 \pmod{2000}$, for $k = 0, \dots, 999$ become friends.
   Also, there are
   $$\binom{2000}{2} = \frac{2000 \cdot 1999}{2} = 1000 \cdot 1999$$
   possible people pairs.
   On the other hand, there are a total $1000 \cdot 2000$ invitations, i.e. 1000 extra invitations over the number of people pairs, so that at least 1000 of the invitations are return invitations.
   Hence the number of pairs of friends cannot be less than 1000.
   Since we have a configuration demonstrating that 1000 pairs of friends is possible and it cannot be fewer, the minimum number of pairs of friends is 1000.