The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

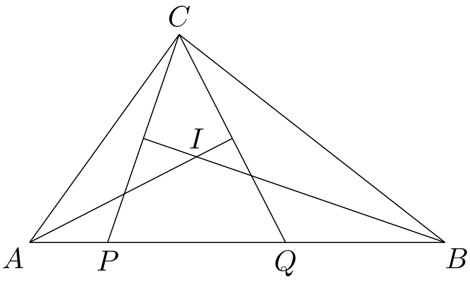
AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2011 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. In $\triangle ABC$, P and Q are points on its longest side AB, such that $AQ = AC$ and $BP = BC$. Prove that the circumcentre of $\triangle CPQ$ coincides with the incentre of $\triangle ABC$. (3 points)

Solution. Let I be the incentre of $\triangle ABC$.
Then IB is the angle bisector of $\angle CBA = \angle CBP$.
Also,



$$\begin{aligned} &BP = BC \\ \therefore\ &\triangle CBP \text{ is isosceles} \\ &\therefore\ IB \text{ is axis of symmetry of } \triangle CBP \\ &\therefore\ IB \text{ is the perpendicular bisector of } CP. \end{aligned}$$

Similarly, IA is the angle bisector of $\angle CBA = \angle CBP$, and

$$\begin{aligned} &AQ = AC \\ \therefore\ &\triangle CAQ \text{ is isosceles} \\ &\therefore\ IA \text{ is axis of symmetry of } \triangle CAQ \\ &\therefore\ IA \text{ is the perpendicular bisector of } CQ. \end{aligned}$$

Thus, I is the point of concurrence of the perpendicular bisectors of $\triangle CPQ$, and hence I is the circumcentre of $\triangle CPQ$.

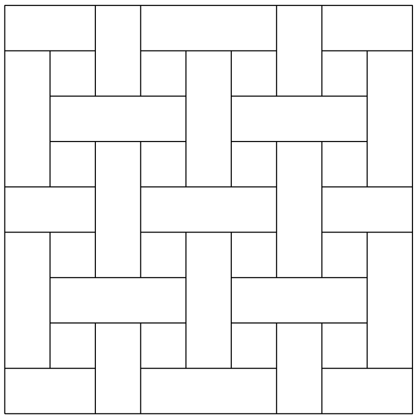
2. Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest.
Prove that not all the berries have been eaten. (4 points)

Solution. Firstly, suppose a guest has eaten an even number of berries. Then proceeding clockwise, whether each guest eats twice as many berries or 6 fewer berries than the next guest, each guest has eaten an even number of berries.
So, for one guest to have eaten an odd number of berries, all guests must have eaten an odd number of berries. But for each guest to maintain parity with the next guest, they must eat 6 fewer berries (twice as many leads to a consumption of an even number of berries). However, if there are N guests seated ("several" implies $N > 0$) at the table, and each guest has eaten 6 fewer berries than the next, then a fixed guest has eaten $6N > 0$ fewer berries than themselves (a contradiction).

Thus all guests have eaten an even number of berries, and hence the total number of berries consumed is even, and hence at least one of the 2011 berries has not been eaten.
Therefore, not all the berries have been eaten.

3. From a 9×9 chessboard, all 16 unit squares whose row numbers and column numbers are both even have been removed.
Dissect the punctured board into rectangular pieces, with as few unit squares as possible. (4 points)

Solution. The minimal number of unit squares is 0, since the diagram shows it is possible to dissect the punctured 9×9 chessboard into rectangular pieces (2×1, 3×1, 1×2, 1×3) with no unit squares.



4. The vertices of a 33-gon are labelled with the integers from 1 to 33. Each edge is then labelled with the sum of the labels of its two vertices.
Is it possible for the edge labels to consist of 33 consecutive numbers? (4 points)

Solution. Suppose we think it's not possible. Then we should look for a contradiction:
Let the vertex labels be $v_1, v_2, \ldots, v_{33}$, and the edge labels be $e_1, e_2, \ldots, e_{33}$. Then

$$\begin{aligned}
e_1 + e_2 + \cdots + e_{33} &= (v_1 + v_2) + (v_2 + v_3) + \cdots + (v_{33} + v_1) \\
&= 2(v_1 + v_2 + \cdots + v_{33}) \\
&= 2(1 + 2 + \cdots + 33) \\
&= 2 \cdot \frac{33+1}{2} \cdot 33 \\
&= \frac{33}{2}(a + \ell), \quad \text{where } a, \ldots, \ell \text{ are the consecutive edge labels in increasing order} \\
\implies \quad a + \ell &= 68 \\
&= 2a + (n-1)d, \quad \text{where } n = 33,\ d = 1 \\
\implies \quad 34 &= a + 16 \\
\implies \quad a &= 18 \text{ (and } \ell = 50).
\end{aligned}$$

So, if it's possible, the consecutive edge labels are $18, 19, \ldots, 50$.
Now, when we try to find a solution with such labels we see that leaving a gap between consecutive vertex labels and interleaving we obtain:

$$17, 1, 18, 2, 19, \ldots, 16, 33, 17$$

(showing the 17 repeating in order to show the complete cycle) so that the edge labels are:

$$18, 19, 20, 21, \ldots, 49, 50.$$

5. On a highway, a pedestrian and a cyclist were going in the same direction, while a cart and a car were coming from the opposite direction. All were travelling at different constant speeds. The cyclist caught up with the pedestrian at 10 o'clock. After a time interval, she met the cart, and after another time interval equal to the first, she met the car. After a third time interval, the car met the pedestrian, and after another time interval equal to the third, the car caught up with the cart.
If the pedestrian met the car at 11 o'clock, when did he meet the cart? (5 points)

Solution. Represent the pedestrian, cyclist, car and cart by p, q, r and s, respectively. Let the first time interval be a and the next different length time interval be b. Then the given information is:

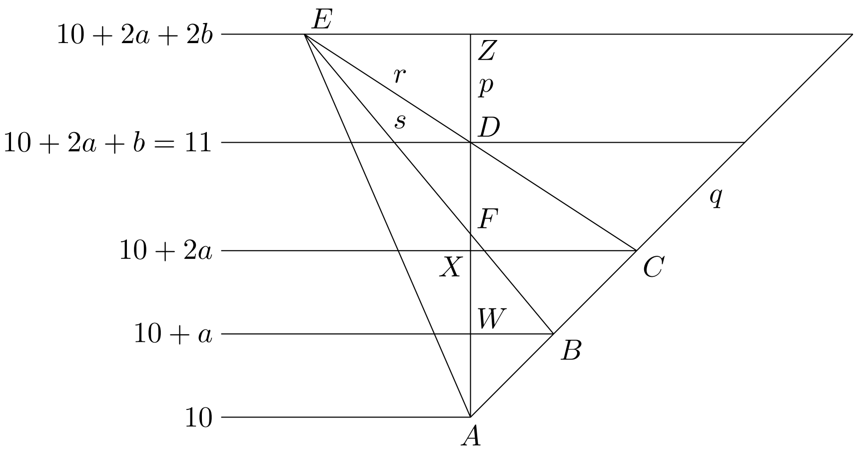
At 10, q meets p.
At $10 + a$, q meets s.
At $10 + 2a$, q meets r.
At $10 + 2a + b = 11$, r meets p.
At $10 + 2a + 2b$, r meets s.
And, we are required to find when s meets p.

Measure all speeds relative to p, or equivalently assume p has 0 speed.
Identify each of p, q, r, s with the line depicting their distance *vs* time graph.
Define the following intersection points of these graphs:

$$p \cap q = A, q \cap s = B, q \cap r = C, r \cap p = D, r \cap s = E, s \cap p = F.$$

Further let the, until now, unnamed intersection points of the horizontal lines $t = 10 + a$, $t = 10 + 2a$ and $t = 10 + 2a + 2b$ (where t is the time variable), with p, be W, X and Z, respectively. Then we obtain the diagram below, via the following steps:

- Choose arbitrary different a and b.
- Draw the horizontal time lines.
- Draw a vertical line representing p.
- Now $t = 10$ meets p at A.
- Draw q of arbitrary positive slope through A.
- Now, $t = 10+2a$ intersects q at C, while $10+2a+b$ intersects p at D, giving $r = CD$.
- And, $10+2a+2b$ intersects r at E, while $t = 10+a$ intersects q at B, giving $s = BE$.



We must find the time corresponding to F.

Consider $\triangle ACE$.

$$\begin{aligned} AW = WX = a \text{ and } WB \parallel XC &\implies AB = AC \\ &\implies s = EB \text{ is a median of } \triangle ACE. \end{aligned}$$

Similarly,

$$\begin{aligned} XD = DZ = b \text{ and } XC \parallel ZE &\implies CD = DE \\ &\implies p = AD \text{ is a median of } \triangle ACE. \end{aligned}$$

Therefore, $F = s \cap p$ is the centroid of $\triangle ACE$, and so $AF = \frac{2}{3}AD$ and $AD = 11 - 10 = 1\,\mathrm{h}$, so that $AF = 40\,\mathrm{min}$.
Thus, the time when s meets p (i.e. the time at which F occurs) is $10:40$.