The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO/TT TRAINING SESSIONS

Tournament of the Towns Problems with Solutions
Junior Paper: Years 8, 9, 10
Northern Autumn 2012 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Five students have first names Clark, Donald, Jack, Robin and Steve, and family names (in a different order) Clarkson, Donaldson, Jackson, Robinson and Stevenson. It is known that

   Clark is 1 year older than Clarkson,
   Donald is 2 years older than Donaldson,
   Jack is 3 years older than Jackson, and
   Robin is 4 years older than Robinson.

   Who is older, Steve or Stevenson and by how much? (3 points)

   Solution. Let c, d, j, r, s be the ages of Clark, Donald, Jack, Robin and Steve, respectively, and let C, D, J, R, S be the ages of Clarkson, Donaldson, Jackson, Robinson and Stevenson respectively. Then, since c, d, j, r, s are just C, D, J, R, S in a different order,

$$\begin{aligned} C + D + J + R + S &= c + d + j + r + s \\ \therefore S - s &= (c - C) + (d - D) + (j - J) + (r - R) \\ &= 1 + 2 + 3 + 4 \\ &= 10. \end{aligned}$$

   So, Stevenson is 10 years older than Steve.

2. Let $C(n)$ be the number of prime divisors of $n \in \mathbb{N}$, e.g. $C(10) = 2$, $C(11) = 1$, $C(12) = 2$. Is the number of pairs of positive integers (a, b) such that $a \neq b$ and

$$C(a + b) = C(a) + C(b)$$

   finite or infinite? (4 points)

   Solution. $S = \{(a, b) \in \mathbb{N}^2 \mid a \neq b, C(a + b) = C(a) + C(b)\}$ is infinite. To prove this we construct an infinite subset of S. Any one of the following subsets T_i will suffice.
   (i) $T_1 = \{(a, b) = (2^t, 2^{t+1}) \mid t \in \mathbb{N}\}$. Here, $C(a) = C(b) = 1$ and

$$\begin{aligned} a + b &= (1 + 2) \cdot 2^t \\ &= 3 \cdot 2^t \\ \implies C(a + b) &= 2 = C(a) + C(b). \end{aligned}$$

   So, $|T_1| = |\mathbb{N}| = \infty$.

(ii) $T_2 = \{(a,b) = (3^t, 3^{t+1}) \mid t \in \mathbb{N}\}$. Here, $C(a) = C(b) = 1$ and

$$\begin{aligned} a + b &= (1+3) \cdot 2^t \\ &= 2^2 \cdot 3^t \\ \implies C(a+b) &= 2 = C(a) + C(b). \end{aligned}$$

So, $|T_2| = |\mathbb{N}| = \infty$.

(iii) $T_3 = \{(a,b) = (p, 5p) \mid 5 < p, p \text{ prime}\}$. Here, $C(a) = 1, C(b) = 2$ and

$$\begin{aligned} a + b &= (1+5)p \\ &= 2 \cdot 3 \cdot p \\ \implies C(a+b) &= 3 = C(a) + C(b). \end{aligned}$$

So, $|T_3|$ is the number of primes greater than 5, which is infinite.

(iv) $T_4 = \{(a,b) = (p, 3^2 \cdot 17 \cdot p) \mid 17 < p, p \text{ prime}\}$. Here, $C(a) = 1, C(b) = 3$ and

$$\begin{aligned} a + b &= (1+153)p \\ &= 2 \cdot 7 \cdot 11 \cdot p \\ \implies C(a+b) &= 4 = C(a) + C(b). \end{aligned}$$

So, $|T_4|$ is the number of primes greater than 17, which is infinite.

3. A 10×10 table is filled out according to the rules of the 'Minesweeper' game: each cell either contains a mine or a number that shows how many mines are in neighbouring cells, where cells are neighbours if they have a common edge or vertex.

   If all mines are removed from the table and then new mines are placed in all previously mine-free cells, with the remaining cells to be filled out with the numbers according to the 'Minesweeper' game rule as above, can the sum of all numbers in the table increase? (5 points)

Solution. We construct a graph as follows:

Represent each cell by a vertex.

Join two vertices by an edge, if their corresponding cells are neighbours.

For brevity, we say *"an edge joins two cells"* if it joins the vertices corresponding to those cells. Now, call an edge

scoring if it joins a mine cell to a vacant cell (the vacant cell will have a number in it corresponding to the number of mines it neighbours), or

non-scoring if it joins two mine cells or two vacant cells.

Observe that the number of *scoring* edges is the total of the numbers in the grid (which we will call the *value* of the grid).

Moreover, the operation of removing the mines and placing new mines at previously mine-free cells, takes scoring edges to scoring edges and non-scoring edges to non-scoring edges.

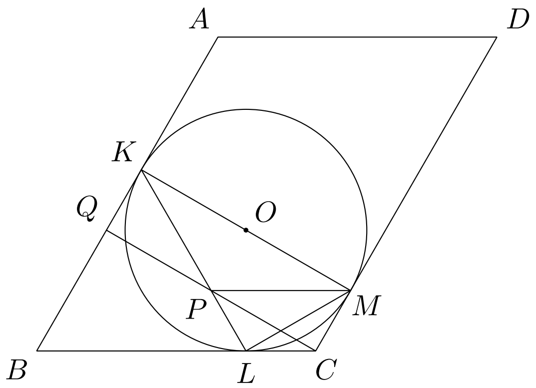
Hence the *value* of the grid is invariant under this operation.

Therefore, the answer is "No, the sum of all numbers in the grid cannot increase."

4. A circle touches sides AB, BC, CD of a parallelogram $ABCD$ at points K, L, M, respectively.

   Prove that the line KL bisects the altitude of $ABCD$ that is dropped to the side AB from C. (5 points)

Solution. Let O be the centre of the circle, Q be the the foot of the perpendicular dropped from C to AB, and $P = KL \cap CQ$.



Since the circle touches AB at K and CD at M, OK and OM are radii of the circle. Now,

$$OK \perp AB,\ OM \perp CD,\ AB \parallel CD$$

$\therefore OK \parallel OM$

$\therefore KOM$ is a straight line

$\therefore KOM$ is a diameter of the circle

Also, $CQ \perp AB$ (and $CQ \perp CD$)

$\therefore CMKQ$ is a rectangle

$\therefore QK = CM.$

$\angle PLM = \angle KLM = 90°$, (angle in a semicircle)

$\angle PCM = \angle QCM = 90°$

$\therefore$ $PLCM$ is a cyclic quadrilateral

$LC = MC$, since LC, MC are tangents to the circle

$\therefore$ $\triangle LCM$ is isosceles

$\therefore$ $\angle KPQ = \angle LPC$, (vertically opposite)

$= \angle LMC$, (angles on arc LC in circumcircle of $PLCM$)

$= \angle MLC$, since $\triangle LCM$ isosceles

$= \angle MPC$, (angles on arc MC in circumcircle of $PLCM$)

$\angle KQP = \angle MCP = 90°$

$QK = CM$, (proved above)

$\therefore$ $\triangle KQP \cong \triangle MCP$, by the AAS Rule

$\therefore$ $QP = CP$.

Hence, KL bisects QC (the altitude from C to AB of $ABCD$).

5. For a class of 20 students several excursions were arranged with at least one student attending each of them.

   Prove that there was an excursion such that each student in that excursion took part in at least 1/20 of all excursions. (5 points)

Solution. Let N be the number of excursions and write $N = 20a + b$ via the "Division Algorithm", except that $a, b \in \mathbb{Z}$ such that $1 \le b \le 20$.

Let $T_1, \dots, T_N$ be the N excusions, and, suppose for a contradiction, that each excursion has a student that has attended $< \frac{1}{20}$ of the excursions. In particular, let s_i be a student attending excursion T_i who has attended $< \frac{1}{20}$ of the excursions. Then $S = \{s_1, \dots, s_N\}$ is a subset of the class of 20 students, with the properties that

(i) For all T_i, there is $s_i \in S$ such that the number of T_j participated in by s_i is $\le a$.

(ii) For all $s_i \in S$, the number of T_j participated in by s_i is $\le a$.

$$\begin{aligned}\therefore N \le \sum\nolimits_{s \in S} a &= |S| \cdot a \\ &\le 20a \\ &< 20a + b = N \text{ (contradiction)}\lightning\end{aligned}$$

Hence, there is as excursion such that each student attending has attended $\ge \frac{1}{20}$ of the excursions.